1. Using differentials, find the approximate value of each of the following:

(a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ (b) $(33)^{-\frac{1}{5}}$

Sol: (a) $y = x^{\frac{1}{4}} \cdots (1)$.

Let Δx be small increment in x and Δy be the corresponding increment in y.

Then $y + \Delta y = (x + \Delta x)^{\frac{1}{4}} \cdots (2)$

Subtract (1) from (2) to get

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}} \Rightarrow (x + \Delta x)^{\frac{1}{4}} = \Delta y + x^{\frac{1}{4}} \Rightarrow (x + \Delta x)^{\frac{1}{4}} = \frac{dy}{dx}\Delta x + x^{\frac{1}{4}}$$

$$(x + \Delta x)^{\frac{1}{4}} = \frac{1}{4x^{\frac{3}{4}}}\Delta x + x^{\frac{1}{4}} \cdots (3)$$

Put $x = \frac{16}{81}$ and $\Delta x = \frac{1}{81}$ in (3) we get

$$\left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right) + \left(\frac{16}{81}\right)^{\frac{1}{4}} \Rightarrow \left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{1}{96} + \frac{2}{3} = \frac{65}{96} \Rightarrow \left(\frac{17}{81}\right)^{\frac{1}{4}} = 0.677$$

(b) Let $y = x^{-\frac{1}{5}} \cdots (1)$.

Let Δx be small increment in x and Δy be the corresponding increment in y.

Then $y + \Delta y = (x + \Delta x)^{-\frac{1}{5}} \cdots (2)$

Subtract (1) from (2) to get

$$\Delta y = (x + \Delta x)^{-\frac{1}{5}} - x^{-\frac{1}{5}} \Rightarrow (x + \Delta x)^{-\frac{1}{5}} = \Delta y + x^{-\frac{1}{5}} \Rightarrow (x + \Delta x)^{-\frac{1}{5}} = \frac{dy}{dx}\Delta x + x^{-\frac{1}{5}}$$

$$\Rightarrow (x + \Delta x)^{-\frac{1}{5}} = -\frac{1}{5x^{\frac{6}{5}}}\Delta x + x^{-\frac{1}{5}} \cdots (3)$$

Put $x = 32$ and $\Delta x = 1$ in (3) we get

$$(33)^{-\frac{1}{5}} = -\frac{1}{5(32)^{\frac{6}{5}}}(1) + (32)^{-\frac{1}{5}} = \frac{-1}{320} + \frac{1}{2} = 0.5 - 0.003 \Rightarrow (33)^{-\frac{1}{5}} = 0.497$$

2. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$

Sol: We have $f(x) = \frac{\log x}{x}, x > 0$

$$\Rightarrow f'(x) = \frac{x\left(\frac{1}{x}\right) - (\log x)(1)}{x^2} = \frac{1-\log x}{x^2}$$

$$\Rightarrow f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1-\log x)(2x)}{x^4} = \frac{-x-2x+2x\log x}{x^4} = \frac{-3x+2x\log x}{x^4} = \frac{x(2\log x - 3)}{x^4} = \frac{2\log x - 3}{x^3}$$

For critical point $f'(x) = 0 \Rightarrow \frac{1-\log x}{x^2} = 0 \Rightarrow \log x = 1 \Rightarrow x = e$

Now $f''(e) = \frac{2\log e - 3}{e^3} = \frac{2-3}{e^3} = -\frac{1}{e^3} < 0$

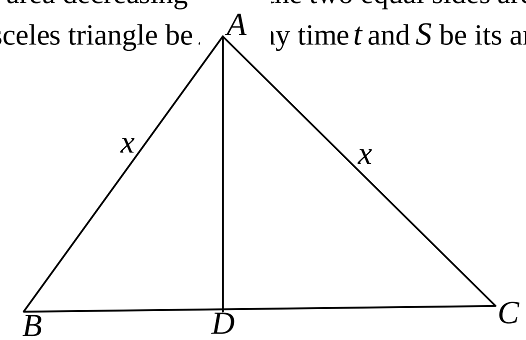
$\Rightarrow f$ has a local maximum at $x = e$

Since f is continuous in the open interval $(0,\infty)$ and has only one extremum at $x = e \in (0,\infty)$,

therefore, $f(x)$ is absolutely maximum at $x = e$.

3. Two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Sol: Let side of the isosceles triangle be x at any time t and S be its area at that time.



Draw $AD \perp BC$.

Then $AD^2 + DC^2 = AC^2 \Rightarrow AD^2 = AC^2 - DC^2 = x^2 - \left(\frac{b}{2}\right)^2 \Rightarrow AD = \sqrt{x^2 - \frac{b^2}{4}}$

$$S = \frac{1}{2}b\sqrt{x^2 - \frac{b^2}{4}} = \frac{b}{4}\sqrt{4x^2 - b^2}$$

$$\frac{dS}{dt} = \frac{dS}{dx}\times\frac{dx}{dt} = \frac{b}{4}\times\frac{1}{2\sqrt{4x^2-b^2}}(8x)\frac{dx}{dt} = \frac{bx}{\sqrt{4x^2-b^2}}\frac{dx}{dt}$$

When $\frac{dx}{dt} = -3$ cm/sec and $x = b$ then $\frac{dS}{dt} = \frac{b\times b}{\sqrt{4b^2-b^2}}\times -3 = \frac{-3b^3}{\sqrt{3}b} = -\sqrt{3}b^2$

Thus $\frac{dS}{dt} = -\sqrt{3}b^2$ cm^2/sec.

4. Find the equation of the normal to the curve $y^2 = 4x$ at the point $(1,2)$

Sol: Given curve is $y^2 = 4x \cdots (1)$

Diff. w.r.t. x we get $2y\frac{dy}{dx}=4 \Rightarrow \frac{dy}{dx}=\frac{2}{y}$

Slope of the tangent to (1) at (1,2) $=\left.\frac{dy}{dx}\right]_{(1,2)}=\frac{2}{2}=1$

$\Rightarrow$ slope of the normal to (1) at (1,2) $=-\frac{1}{1}=-1$

Thus equation of normal to (1) at (1,2) is $y-2=-1(x-1) \Rightarrow x+y-3=0$

5.Show that the normal at any point θ to the curve $x=a\cos\theta+a\theta\sin\theta, y=a\sin\theta-a\theta\cos\theta$ is at a constant distance from the origin.

Sol: The given curve is $x=a\cos\theta+a\theta\sin\theta, y=a\sin\theta-a\theta\cos\theta$

$$\Rightarrow \frac{dx}{d\theta}=-a\sin\theta+a\left(\theta\cos\theta+\sin\theta\right)=a\theta\cos\theta$$

and $\frac{dy}{d\theta}=a\cos\theta-a\left(-\theta\sin\theta+\cos\theta\right)=a\theta\sin\theta$

Therefore, slope of the tangent at $'\theta'=\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{a\theta\sin\theta}{a\theta\cos\theta}=\tan\theta$

Thus, slope of the normal at $'\theta'=-\frac{1}{\frac{dy}{dx}}=-\frac{1}{\tan\theta}=-\cot\theta$

Hence the equation of the normal at $'\theta'$ is

$$y-\left(a\sin\theta-a\theta\cos\theta\right)=-\frac{\cos\theta}{\sin\theta}\left(x-\left(a\cos\theta+a\theta\sin\theta\right)\right)$$

$$\Rightarrow y\sin\theta-a\sin^2\theta+a\theta\sin\theta\cos\theta=-x\cos\theta+a\cos^2\theta+a\theta\sin\theta\cos\theta$$

$$\Rightarrow x\cos\theta+y\sin\theta=a\left(\sin^2\theta+\cos^2\theta\right)$$

$$\Rightarrow x\cos\theta+y\sin\theta=a$$

Compare with the normal form of the line $x\cos\omega+y\sin\omega=p$, we get $p=a$

Thus normal to the given curve at any point $'\theta'$ is at a constant distance $'a'$ from origin.

6. Find the intervals in which the function f given by $f\left(x\right)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$ is

(i) increasing (ii) decreasing

Solution: $f\left(x\right)=\frac{4\sin x-2x-x\cos x}{2+\cos x}=\frac{4\sin x-x(2+\cos x)}{2+\cos x}=\frac{4\sin x}{2+\cos x}-x$

$$\Rightarrow f'\left(x\right)=4\left\{\frac{(2+\cos x)\cos x-\sin x(-\sin x)}{(2+\cos x)^2}\right\}-1$$

$$\Rightarrow f'\left(x\right)=4\left\{\frac{2\cos x+\cos^2 x+\sin^2 x}{\left(2+\cos x\right)^2}\right\}-1=4\left\{\frac{2\cos x+1}{\left(2+\cos x\right)^2}\right\}-1$$

$$\Rightarrow f'(x)=\frac{8\cos x+4-(2+\cos x)^2}{(2+\cos x)^2}=\frac{8\cos x+4-4-\cos^2 x-4\cos x}{(2+\cos x)^2}$$

$$\Rightarrow f'(x)=\frac{4\cos x-\cos^2 x}{(2+\cos x)^2}=\frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

(i) $f(x)$ is increasing if $f'(x)>0 \Rightarrow \frac{\cos x(4-\cos x)}{(2+\cos x)^2}>0$

As $(2+\cos x)^2>0$ & $4-\cos x>0\ \forall x\in R$

Thus $f'(x)\geq 0 \Rightarrow \cos x>0 \Rightarrow x\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right)$

(ii) $f(x)$ is decreasing if $f'(x)<0 \Rightarrow \frac{\cos x(4-\cos x)}{(2+\cos x)^2}<0 \Rightarrow \cos x<0 \Rightarrow x\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$

7. Find the intervals in which the function f given by $f(x)=x^3+\frac{1}{x^3}, x\neq 0$

Sol: $f(x)=x^3+\frac{1}{x^3} \Rightarrow f'(x)=3x^2-\frac{3}{x^4}$

$f(x)$ is increasing if

$f'(x)>0 \Rightarrow 3x^2-\frac{3}{x^4}>0 \Rightarrow \frac{3x^6-3}{x^4}>0 \Rightarrow 3(x^6-1)>0 \Rightarrow x^6>1 \Rightarrow x<-1$ or $x>1$

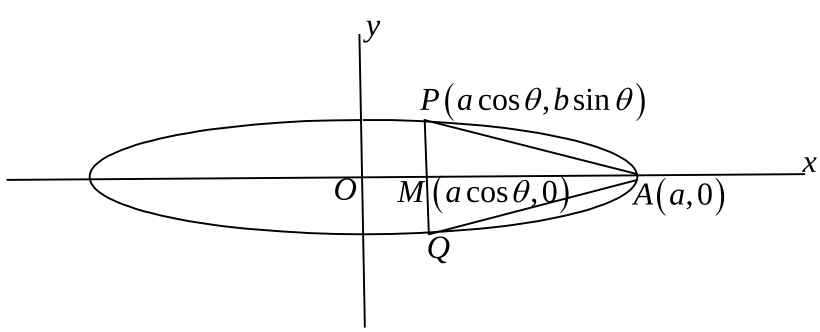
$f(x)$ is decreasing if $f'(x)<0 \Rightarrow 3x^2-\frac{3}{x^4}<0 \Rightarrow x^6<1 \Rightarrow -1<x<1$

But as $x\neq 0$, therefore f is decreasing in $(-1,0)\cup(0,1)$

Thus f is increasing in $(-\infty,-1)\cup(1,\infty)$ and decreasing in $(-1,0)\cup(0,1)$

8. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.

Sol:



Any point on P on the ellipse is $\left(a\cos\theta, b\sin\theta\right)$. From P draw PM $\perp$ OX and produce to meet the ellipse at Q, then APQ is isosceles triangle, let S be its area, then

$$S=\frac{1}{2}AM\times PQ=\frac{1}{2}\times\left(OA-OM\right)\times 2PM=\left(a-a\cos\theta\right)b\sin\theta, 0<\theta<\pi$$

$$S=ab\left(\sin\theta-\sin\theta\cos\theta\right)=ab\left(\sin\theta-\frac{1}{2}\sin 2\theta\right)$$

$$\Rightarrow\frac{dS}{d\theta}=ab\left(\cos\theta-\cos 2\theta\right)\text{ and }\frac{d^2S}{d\theta^2}=ab\left(-\sin\theta+2\sin 2\theta\right)$$

For maxima/minima

$$\frac{dS}{d\theta}=0\Rightarrow ab\left(\cos\theta-\cos 2\theta\right)=0\Rightarrow\cos 2\theta=\cos\theta\Rightarrow 2\theta=2\pi-\theta\Rightarrow 3\theta=2\pi\Rightarrow\theta=\frac{2\pi}{3}$$

For $\theta=\frac{2\pi}{3}$, we have $\frac{d^2S}{d\theta^2}=ab\left(-\sin\frac{2\pi}{3}+2\sin\frac{4\pi}{3}\right)<0$

Thus S is maximum when $\theta=\frac{2\pi}{3}$

And maximum value of

$$S=ab\left(\sin\frac{2\pi}{3}-\frac{1}{2}\sin\frac{4\pi}{3}\right)=ab\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)\right)=\frac{3\sqrt{3}}{4}ab\text{ sq units}.$$

9. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2m and volume is8 m^3. If building of tank costs Rs 70 per sq metres for the base Rs 45 per square meter for sides. What is the cost of least expensive tank?

Sol: Let length and breadth of the tank be x m and y m.

Thus $V=8\Rightarrow 2xy=8$ (As depth is 2m) $\Rightarrow y=\frac{4}{x}$

Area of the base is xy m^2. Its cost is Rs $70xy=70x\frac{4}{x}=280$. Area of 4 walls of the tank is

$2\left(l+b\right)h=2\left(x+y\right)\times 2=4\left(x+y\right)=4\left(x+\frac{4}{x}\right)$ and its cost is Rs

$$45\times 4\left(x+\frac{1}{x}\right)=180\left(x+\frac{4}{x}\right)$$

Let C be the total cost.

Thus $C=280+180\left(x+\frac{4}{x}\right)$

$$\Rightarrow\frac{dC}{dx}=180\left(1-\frac{4}{x^2}\right)\ \&\ \frac{dC^2}{dx^2}=180\left(\frac{8}{x^3}\right)$$

For maxima/minima, we have $\frac{dC}{dx}=0\Rightarrow 180\left(1-\frac{4}{x^2}\right)\Rightarrow x^2=4\Rightarrow x=2$ (As $x>0$)

$$\left.\frac{dC^2}{dx^2}\right]_{x=2} = 180\left(\frac{8}{2^3}\right) > 0$$

Thus cost is least when $x = 2$ and least expensive cost is $S = 280 + 180\left(2 + \frac{4}{2}\right) = \text{Rs}1000$

10. The sum of parameter of a circle and a square is k, where k is some constant. Prove that the sum of the areas is the least when the side of the square is double the radius of the circle.

Sol. Let r be the radius of the circle and x be the side of the square then $2\pi r + 4x = k$

$$\Rightarrow x = \frac{k - 2\pi r}{4} \cdots (1)$$

Let S be the sum of areas of the circle and square, then

$$S = \pi r^2 + x^2 = \pi r^2 + \left(\frac{k - 2\pi r}{4}\right)^2 = \pi r^2 + \frac{k^2}{16} + \frac{\pi^2 r^2}{4} - \frac{k\pi r}{4}$$

$$\frac{dS}{dr} = 2\pi r + \frac{2\pi^2 r}{4} - \frac{k\pi}{4} \;\&\; \frac{d^2S}{dr^2} = 2\pi + \frac{2\pi^2}{4} > 0 \quad \forall r$$

Now $\frac{dS}{dr} = 0 \Rightarrow 2\pi r + \frac{2\pi^2 r}{4} - \frac{k\pi}{4} = 0 \Rightarrow r\left(2\pi + \frac{\pi^2}{2}\right) = \frac{k\pi}{4}$

$$\Rightarrow r\left(\frac{4\pi + \pi^2}{2}\right) = \frac{k\pi}{4} \Rightarrow r = \frac{k}{2(4+\pi)}$$

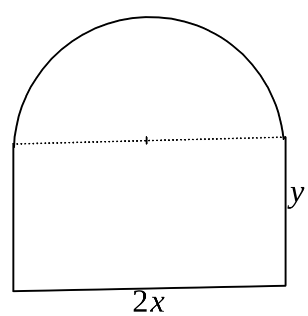
As for this r, we have $\frac{d^2S}{dr^2} > 0$, therefore, S is least when $r = \frac{k}{2(4+\pi)}$ and then

$$x = \frac{1}{4}\left[k - \frac{2\pi}{2}\left(\frac{k}{4+\pi}\right)\right] = \frac{k}{4+\pi}$$

Hence S is least when the side of the square is double the radius of the circle.

11. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Sol:



Let length and breadth of the rectangle are $2x$ m and y m respectively. Then radius of the semicircular opening is x m.

Thus Perimeter $=10\text{ m} \Rightarrow 2x+2y+\pi x=10 \Rightarrow (2+\pi)x+2y=10 \Rightarrow y=\dfrac{10-(2+\pi)x}{2}$

Let A be the area of the window. Window will admit maximum light if area of the window is maximum. Thus we have to maximize A.

$$A=2xy+\frac{\pi x^2}{2}=2x\left[\frac{10-(2+\pi)x}{2}\right]+\frac{\pi x^2}{2} \Rightarrow A=10x-(2+\pi)x^2+\frac{\pi x^2}{2}$$

$$A=10x-2x^2-\frac{\pi x^2}{2}$$

$$\frac{dA}{dx}=10-4x-\pi x \;\&\; \frac{dA^2}{dx^2}=-4-\pi<0\,\forall x$$

For maxima and minima $\dfrac{dA}{dx}=0 \Rightarrow 10-4x-\pi x=0 \Rightarrow x=\dfrac{10}{4+\pi}$

For this value of x we have $\dfrac{dA^2}{dx^2}<0$.

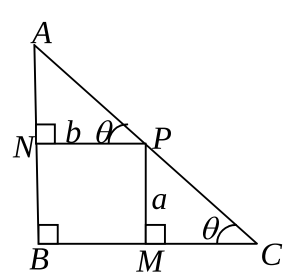
Thus area is minimum when

$$x=\frac{10}{4+\pi}\text{ and therefore } y=\frac{10-(2+\pi)\times\left(\frac{10}{4+\pi}\right)}{2}=\frac{40+10\pi-20-10\pi}{2(4+\pi)}=\frac{10}{4+\pi}$$

Thus in order to admit the maximum light length $=\dfrac{20}{4+\pi}$ m and breadth $=\dfrac{20}{4+\pi}$ m

12. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{3}}$

Sol: Let ABC be the given triangle such that $\angle B=90°$. Let P be a point on hypotenuse AC. Draw $PM \perp BC$ and $PN \perp AB$. Let $PM=a$ and $PN=b$. Also let $\angle C=\theta$. Thus $\angle APN=\theta$. Then in triangle PMC we have $\operatorname{cosec}\theta=\dfrac{PC}{PM}=\dfrac{PC}{a} \Rightarrow PC=a\operatorname{cosec}\theta$. In triangle APN, $\sec\theta=\dfrac{AP}{PN}=\dfrac{AP}{b} \Rightarrow AP=b\sec\theta$. Let y be the length of the hypotenuse.



Thus $y=a\operatorname{cosec}\theta+b\sec\theta$

$$\frac{dy}{d\theta} = -a\operatorname{cosec}\theta\cot\theta + b\sec\theta\tan\theta = -\frac{a\cos\theta}{\sin^2\theta} + \frac{b\sin\theta}{\cos^2\theta} = \frac{-a\cos^3\theta + b\sin^3\theta}{\sin^2\theta\cos^2\theta}$$

$$\frac{d^2y}{d\theta^2} = a\operatorname{cosec}^3\theta + a\operatorname{cosec}\theta\cot^2\theta + b\sec^3\theta + b\sec\theta\tan^2\theta$$

As $\theta < 90^\circ$ & $a, b > 0 \Rightarrow \frac{d^2y}{d\theta^2} > 0$

For max/ min we have

$$\frac{dy}{d\theta} = 0 \Rightarrow \frac{-a\cos^3\theta + b\sin^3\theta}{\sin^2\theta\cos^2\theta} = 0 \Rightarrow b\sin^3\theta = a\cos^3\theta \Rightarrow \tan^3\theta = \frac{a}{b} \Rightarrow \tan\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$$

For this value of $\frac{d^2y}{d\theta^2} > 0$

Thus y is minimum when $\tan\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}} = \frac{a^{1/3}}{b^{1/3}} \Rightarrow \sin\theta = \frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$ & $\cos\theta = \frac{b1}{\sqrt{a^{2/3}+b^{2/3}}}$

Thus minimum length of hypotenuse $= a\operatorname{cosec}\theta + b\sec\theta = \frac{a\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} + \frac{b\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}$

$$= \sqrt{a^{2/3}+b^{2/3}}\left(a^{2/3}+b^{2/3}\right) = \left(a^{2/3}+b^{2/3}\right)^{3/2}$$

13. Find the points at which the function f given by $f(x) = (x-2)^4(x+1)^3$ has

(i) local maxima (ii) local minima (iii) point of inflexion

Sol: $f(x) = (x-2)^4(x+1)^3$

$$f'(x) = 4(x-2)^3(x+1)^3 + 3(x+1)^2(x-2)^4$$

$$\Rightarrow f'(x) = (x-2)^3(x+1)^2\{4(x+1) + 3(x-2)\} = (x-2)^3(x+1)^2(7x-2)$$

For critical points $f'(x) = 0 \Rightarrow x = 2, x = -1, x = \frac{2}{7}$

For $x < -1,\ f'(x) > 0$

For $-1 < x < \frac{2}{7},\ f'(x) > 0$

For $\frac{2}{7} < x < 2,\ f'(x) < 0$

For $x > 2,\ f'(x) > 0$

Thus f has local maxima at $x = \frac{2}{7}$, local minima at $x = 2$ and $x = -1$ is point of inflexion.

14. Find the absolute maximum and minimum values of the function f given by $f(x) = \cos^2 x + \sin x, x \in [0, \pi]$

Sol: $f(x) = \cos^2 x + \sin x, 0 \le x \le \pi$

$$\Rightarrow f'(x) = -2\cos x\sin x + \cos x = \cos x(-2\sin x + 1), 0 < x < \pi$$

For critical point $f'(x)=0 \Rightarrow \cos x = 0$ or $\sin x = \frac{1}{2}$. As $0 < x < \pi$

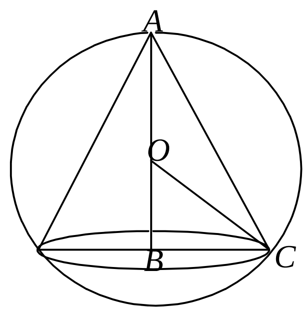
$\Rightarrow x = \frac{\pi}{2}, x = \frac{\pi}{6}, x = \frac{5\pi}{6}$

Now $f(0)=1,\ f(\pi)=1,\ f\left(\frac{\pi}{2}\right)=1,\ f\left(\frac{\pi}{6}\right)=\frac{5}{4},\ f\left(\frac{5\pi}{6}\right)=\frac{5}{4}$

Thus absolute minimum value is 1 and absolute maximum value is $\frac{5}{4}$

15. Show that the altitude right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Sol:



Let x be radius of the base of the cone and y be its height. Let V be its volume.

$OB = AB - OA = y - r$.

$OB^2 + BC^2 = OC^2 \Rightarrow (y-r)^2 + x^2 = r^2 \Rightarrow y^2 - 2yr + r^2 + x^2 = r^2 \Rightarrow x^2 = 2yr - y^2$

Thus $V = \frac{1}{3}\pi x^2 y = \frac{\pi}{3}\left(2yr - y^2\right)y \Rightarrow V = \frac{\pi}{3}\left(2y^2 r - y^3\right)$

$\frac{dV}{dy} = \frac{\pi}{3}\left(4yr - 3y^2\right)$ and $\frac{d^2V}{dy^2} = \frac{\pi}{3}(4r - 6y)$

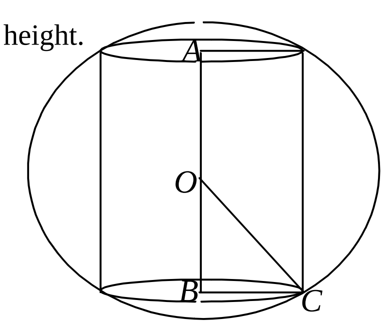
For max/min, we have $\frac{dV}{dy} = 0 \Rightarrow \frac{\pi}{3}\left(4yr - 3y^2\right) = 0 \Rightarrow y(4r - 3y) = 0$. As $y \neq 0$ therefore $y = \frac{4r}{3}$

$\left.\frac{dV^2}{dy^2}\right]_{y=\frac{4r}{3}} = \frac{\pi}{3}(4r - 8r) = -\frac{4\pi r}{3} < 0$

Thus volume of cone is maximum when its height is $\frac{4r}{3}$.

17. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{4R}{3}$.

Sol: Let x be radius of the base of the cylinder and y be its height.



Then $AB = y \Rightarrow OB = \frac{y}{2}$ Now $OB^2 + BC^2 = OC^2 \Rightarrow \frac{y^2}{4} + x^2 = R^2 \Rightarrow x^2 = R^2 - \frac{y^2}{4}$

Let V be the volume of the cylinder, then $V = \pi x^2 y = \pi\left(R^2 - \frac{y^2}{4}\right)y = \pi\left(R^2 y - \frac{y^3}{4}\right)$

This $\frac{dV}{dy} = \pi\left(R^2 - \frac{3y^2}{4}\right)$ and $\frac{d^2V}{dy^2} = \pi\left(-\frac{6y}{4}\right) = -\frac{3\pi y}{2}$

For max/min, $\frac{dV}{dy} = 0 \Rightarrow \pi\left(R^2 - \frac{3y^2}{4}\right) = 0 \Rightarrow 3y^2 = 4R^2 \Rightarrow y^2 = \frac{4R^2}{3} \Rightarrow y = \pm\frac{2R}{\sqrt{3}}$.

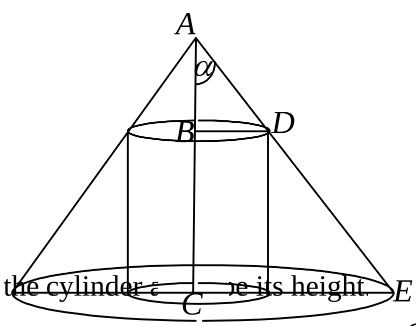
But as height can not be negative, we have $y = \frac{2R}{\sqrt{3}}$

Now $\left.\frac{d^2V}{dy^2}\right]_{y=\frac{2R}{\sqrt{3}}} = -\frac{3\pi}{2} \times \frac{2R}{\sqrt{3}} < 0$

Thus volume of the cylinder is maximum if its height is $\frac{2R}{\sqrt{3}}$

18. Show that the height of the cylinder of greatest volume that can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of the cylinder is $\frac{4}{27}\pi h^3 \tan^2\alpha$.

Sol:



Let x be the radius of the cylinder and y be its height.

Now $AB = AC - BC \Rightarrow AB = h - y$. Now in ΔACE, $\frac{CE}{AC} = \tan\alpha \Rightarrow \frac{CE}{h} = \tan\alpha \Rightarrow CE = h\tan\alpha$

As $\Delta ACE \sim \Delta ABD$, we have $\frac{AB}{AC} = \frac{BD}{CE} \Rightarrow \frac{h-y}{h} = \frac{x}{h\tan\alpha} \Rightarrow h - y = \frac{x}{\tan\alpha} \Rightarrow y = h - \frac{x}{\tan\alpha}$

Let V be the volume of the cylinder.

$V = \pi x^2 y = \pi x^2\left(h - \frac{x}{\tan\alpha}\right) \Rightarrow V = \pi\left(hx^2 - \frac{x^3}{\tan\alpha}\right)$

$\frac{dV}{dx} = \pi\left(2hx - \frac{3x^2}{\tan\alpha}\right)$ and $\frac{dV^2}{dx^2} = \pi\left(2h - \frac{6x}{\tan\alpha}\right)$

For max/min, we have $\frac{dV}{dx} = 0 \Rightarrow \pi\left(2hx - \frac{3x^2}{\tan\alpha}\right) \Rightarrow x\left(2h - \frac{3x}{\tan\alpha}\right) = 0$

But x being the radius of the cylinder, therefore, $x \neq 0 \Rightarrow x = \frac{2h\tan\alpha}{3}$

Now for this value of x we have, $\frac{dV^2}{dx^2} = \pi\left(2h - \frac{6}{\tan\alpha} \times \frac{2h\tan\alpha}{3}\right) = \pi\left(2h - 4h\right) = -2\pi h < 0$

Thus the volume of the cylinder is maximum when $x = \frac{2h\tan\alpha}{3}$

Now $y = h - \frac{x}{\tan\alpha} = h - \frac{2h}{3} = \frac{h}{3}$

Thus the height of the cylinder of maximum volume is 1 3rd of the height of the cone and maximum volume is given by $V = \pi x^2 y = \pi\left(\frac{2h\tan\alpha}{3}\right)^2\left(\frac{h}{3}\right) = \frac{4}{27}\pi h^3 \tan^2\alpha$