Continuity and Differentiability

Continuity at a point

Definition: A function f is said to be continuous at $x=a$ if $\lim_{x\to a} f(x)=f(a)$ or $\lim_{x\to a+} f(x)=\lim_{x\to a-} f(x)=f(a)$

Example-1: Show that the function f given by $f(x)=\begin{cases} x+1 \text{ if } x\le 1 \\ 2x \text{ if } x>1 \end{cases}$

Solution: $\text{LHL (at } x=1)=\lim_{x\to 1-} f(x)=\lim_{x\to 1}(x+1)=1+1=2$ $\quad (\text{as for } x<1, f(x)=x+1)$

$\text{RHL(at } x=1)=\lim_{x\to 1+} f(x)=\lim_{x\to 1}(2x)=2\times 1=2$ $\quad (\text{as for } x>1, f(x)=2x)$

Thus $\lim_{x\to 1} f(x)$ exists and $\lim_{x\to 1} f(x)=2$.

Now $f(1)=1+1=2$ $\quad$ (As for $x\le 1, f(x)=x+1$)

Thus $\lim_{x\to 1} f(x)=f(1)$

Hence f is continuous at $x=1$

Example-2: Prove that the function f given by $f(x)=\begin{cases} \dfrac{\sin x}{x} \text{ if } x<0 \\ x+1 \text{ if } x\ge 0 \end{cases}$ is continuous at $x=0$

Solution: $\text{LHL (at } x=0)=\lim_{x\to 0-} f(x)=\lim_{x\to 0}\dfrac{\sin x}{x}=1$

$\text{RHL (at } x=0)=\lim_{x\to 0^+} f(x)=\lim_{x\to 0}(x+1)=1$

Thus $\lim_{x\to 0} f(x)$ exists and $\lim_{x\to 0} f(x)=1$.

Now $f(0)=0+1=1$

Thus $\lim_{x\to 0} f(x)=f(0)$. Hence f is continuous at $x=0$

Example-3: Discuss the continuity of the function f defined as $f(x)=\begin{cases} \dfrac{x}{|x|} \text{ if } x\ne 0 \\ 1 \text{ if } x=0 \end{cases}$ at $x=0$

Solution: $\text{LHL (at } x=0)=\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\dfrac{|x|}{x}=\lim_{x\to 0}-\left(\dfrac{x}{x}\right)=\lim_{x\to 0}(-1)=-1$

$\text{RHL (at } x=0)=\lim_{x\to 0+} f(x)=\lim_{x\to 0+}\dfrac{|x|}{x}=\lim_{x\to 0}\left(\dfrac{x}{x}\right)=\lim_{x\to 0}(1)=1$

As $\text{LHL}\ne\text{RHL}$ therefore $\lim_{x\to 0} f(x)$ does not exists and hence $f(x)$ is discontinuous at x=0.

Example-4: Discuss the continuity of the function f defined as $f(x)=\begin{cases} 2x+3, x\ge 1 \\ 3x-1, x<1 \end{cases}$ at $x=1$

Solution: $(\text{LHL at } x=1)=\lim_{x\to 1-} f(x)=\lim_{x\to 1}(3x-1)=3(1)-1=2$

$(\text{RHL at } x=1)=\lim_{x\to 1+} f(x)=\lim_{x\to 1} 2(1)+3=5$

As $\lim_{x\to 1-} f(x)\ne\lim_{x\to 1+} f(x)$. Thus f is continuous at $x=1$.

Example-4: Discuss the continuity of the function f defined as $f(x)=\begin{cases} \dfrac{x^2-4}{x-2}, x\ne 2 \\ 4, x=2 \end{cases}$

Solution: $\lim_{x\to 2-} f(x)=\lim_{x\to 2}\dfrac{x^2-4}{x-2}=\lim_{x\to 2}\dfrac{(x-2)(x+2)}{(x-2)}=\lim_{x\to}(x+2)=2+2=4$

$$\lim_{x\to 2+} f(x)=\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}=\lim_{x\to}(x+2)=2+2=4$$

As $\lim_{x\to 2-} f(x)=\lim_{x\to 2+} f(x)=4$ therefore $\lim_{x\to 2} f(x)$ exists and $\lim_{x\to 2} f(x)=4$

Also $f(2)=4$. Thus $\lim_{x\to 2} f(x)=f(2)$, therefore f is continuous at $x=2$

Example-5: Discuss the continuity of the function f de3fined as $f(x)=\begin{cases}\frac{\sqrt{1-\cos 2x}}{x}, x\neq 0\\ 1, x=0\end{cases}$

Solution: $\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\frac{\sqrt{1-\cos 2x}}{x}=\lim_{x\to 0-}\frac{\sqrt{2\sin^2 x}}{x}=\lim_{x\to 0-}\frac{\sqrt{2}|\sin x|}{x}=\lim_{x\to 0}\frac{-\sqrt{2}\sin x}{x}=-\sqrt{2}$

$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}\frac{\sqrt{1-\cos 2x}}{x}=\lim_{x\to 0+}\frac{\sqrt{2\sin^2 x}}{x}=\lim_{x\to 0+}\frac{\sqrt{2}|\sin x|}{x}=\lim_{x\to 0}\frac{\sqrt{2}\sin x}{x}=\sqrt{2}$$

Thus $\lim_{x\to 0^+} f(x)\neq\lim_{x\to 0^+} f(x)$

Therefore $\lim_{x\to 0} f(x)$ does not exist.

Hence function f is discontinuous at $x=0$

Example-5: Find the value of λ such that the function f defined as $f(x)=\begin{cases}3x^2+\lambda, x\leq 1\\ 2x, x>1\end{cases}$

Solution: $\lim_{x\to 1-} f(x)=\lim_{x\to 1}(3x^2+\lambda)=3+\lambda$, $\lim_{x\to 1+} f(x)=\lim_{x\to 1}(2\text{x})=2$, $f(1)=3(1)^2+\lambda=3+\lambda$

As f is continuous at $x=1$

therefore $\lim_{x\to 1-} f(x)=\lim_{x\to +} f(x)=f(1)$

$\Rightarrow 3+\lambda=2\Rightarrow\lambda=-1$

Example-6: Find the value of a and b such that the function $\text{f}(\text{x})=\begin{cases}5, \text{x}\leqslant 2\\ \text{ax}+\text{b}, 2<\text{x}<10\\ 21, \text{x}\geqslant 10\end{cases}$ such that f is continuous at $\text{x}=2$ and $\text{x}=10$

Solution: Since the function is continuous at $\text{x}=2$ therefore we have

$\lim_{x\to 2-} f(x)=\lim_{x\to 2+} f(x)=f(2)$

$\Rightarrow\lim_{x\to 2}(5)=\lim_{x\to 2}(ax+b)=5\Rightarrow 2a+b=5\cdots(1)$

Since the function is continuous at $x=10$ therefore

$\lim_{x\to 10-} f(x)=\lim_{x\to 10+} f(x)=f(10)$

$\Rightarrow\lim_{x\to 10}(ax+b)=\lim_{x\to 10}(21)=21\Rightarrow 10a+b=21\cdots(2)$

Solving (1) and (2) we get $x=2, y=1$

Example-7: Show that the function f given by $f(x)=\begin{cases}\frac{\sin x}{x}+\cos x, x\neq 0\\ 2, x=0\end{cases}$ is continuous at x=0

Solution: $\lim_{x\to 0-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\frac{\sin(-h)}{-h}+\cos(-h)=\lim_{h\to 0}\frac{\sin h}{h}+\lim_{h\to 0}\cos h=1+1=2$

$$\lim_{x\to 0+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\frac{\sin(0+h)}{h}+\cos(0+h)=\lim_{h\to 0}\frac{\sin h}{h}+\lim_{h\to 0}\cos h=1+1=2$$

Therefore $\lim_{x\to 0} f(x)=2$. Also $f(0)=2$. Thus $\lim_{x\to 0} f(x)=f(2)$.

Hence f is continuous at $x=2$

Exercise 5.1

1. Test the continuity of the function f defined as $f(x)=\begin{cases} x^2-1, \text{if } x\le 0, \\ 2\text{x}-1, \text{if } x>0 \end{cases}$ at $x=0$

2. A function f is defined as $f(x)=\begin{cases} \dfrac{x^2-9}{x-3}, \text{if } x\neq 3 \\ 6, \text{if } x=3 \end{cases}$. Show that f is continuous at $x=3$

3. Show that the function f defined as $f(x)=\begin{cases} \dfrac{\sin x}{x}, \text{if } x<0 \\ x+1, \text{if } x\ge 0 \end{cases}$ is continuous at $x=0$

4. Show that the function f defined ad $f(x)=\begin{cases} \dfrac{1-\cos x}{x}, \text{if } x\neq 0 \\ 1, \text{if } x=0 \end{cases}$ is discontinuous at $x=0$

5. Discuss the continuity of the function f defined as $f(x)=\begin{cases} \dfrac{\sin 5\text{x}}{x}, \text{if } x\neq 0 \\ 3, \text{if } x=0 \end{cases}$ at $x=0$

6. Discuss the continuity of the function $f(x)=\begin{cases} \dfrac{|x^2-1|}{x-1}, \text{if } x\neq 1 \\ 2, \text{if } x=1 \end{cases}$ at $x=1$

7. Discuss the continuity of the function f defined as $f(x)=\begin{cases} x^3-1, \text{if } x\, x\le 0 \\ 3\text{x}^2, \text{if } x>0 \end{cases}$ $at\, x=0$

8. Discuss the continuity of the function f defined as $f(x)=\begin{cases} \dfrac{x-|x|}{2}, \text{if } x\neq 0 \\ 2, \text{if } x=0 \end{cases}$ at $x=0$

9. Find the value of k such that the function f defined as $f(x)=\begin{cases} \dfrac{\sin 3x}{x}, \text{if } x\neq 0 \\ k, \text{if } x=0 \end{cases}$ is continuous at $x=0$

10. Find the value of λ such that the function f defined as $f(x)=\begin{cases} \lambda(x^2-1), \text{if } x\le 0 \\ 2x, \text{if } x>0 \end{cases}$ is continuous at $x=0$

11. Find the value of k so that the function f defined as $f(x)=\begin{cases} kx+3, \text{if } x\le 2 \\ 2x-3, \text{if } x>2 \end{cases}$ is continuous at $x=2$.

12. Find the values of a and b such that the function f defined as $f(x)=\begin{cases} 5, \text{if } x\le 2 \\ ax+b, \text{if } 2<x<10 \\ 21, \text{if } x\ge 10 \end{cases}$ is continuous at $x=2$ and $x=5$

13. Find the value of k such that the function f defined as $f(x)=\begin{cases} \dfrac{k\cos x}{\pi-2x}, \text{if } x\neq \dfrac{\pi}{2} \\ 3, \text{if } x=\dfrac{\pi}{2} \end{cases}$ is continuous at $x=\dfrac{\pi}{2}$

Continuity on an interval

Definition: A function f is said to be continuous in open interval (a, b) if it is continuous at each point of the interval.

Definition: A function f is said to be continuous in closed interval $[a, b]$ if and only if

(i) it is continuous in open interval (a, b)

(ii) $\lim_{x \to a+} f(x) = f(a)$ and $\lim_{x \to b-} f(x) = f(b)$

Continuous Function: A function is said to be continuous if it is continuous in its domain.

Every where Continuous Function: A function is said to be continuous everywhere if it is continuous in $(-\infty, \infty)$

Results: Let f and g be two continuous functions on their common domain D, then

(i) $f+g$ is continuous on D

(ii) $f-g$ is continuous on D

(iii) fg is continuous on D

(iv) kf is continuous on D, where k is any constant.

(v) $\frac{f}{g}$ is continuous on $D - \{x : g(x) = 0\}$

(vi) $\frac{1}{f}$ is continuous on $D - \{x : f(x) = 0\}$

Result: Composition of two continuous functions is a continuous function.

Result: If f is a continuous function on D then $|f|$ is continuous on D.

Result:

(i) Constant function is continuous everywhere

(ii) A polynomial function is continuous everywhere.

(iii) Every rational function is continuous at every point of its domain.

(iv) The exponential function $a^x, a > 0$

(v) The function e^x is everywhere continuous.

(vi) $\sin x$ is everywhere continuous.

(vi) $\cos x$ cosine function is everywhere continuous.

(vii) $\tan x, \cot x, \sec x, \operatorname{cosec} x$ are continuous in their domain.

(viii) $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \sec^{-1} x, \operatorname{cosec}^{-1} x, \cot^{-1} x$ are continuous in their domain.

Example-1: Show that the function f defined as $f(x) = \begin{cases} x^2+5 & \text{for } x \le 1 \\ 2x+4 & \text{for } x > 1 \end{cases}$

Solution: For $x < 1, f(x) = x^2 + 5$, which is continuous being a polynomial function.

For $x > 1, f(x) = 2x + 4$, which is continuous being a polynomial function.

Let us now consider the continuity of f at $x = 1$

$\lim_{x \to 1-} f(x) = \lim_{x \to 1} x^2 + 5 = 1 + 5 = 6$

$\lim_{x \to 1+} f(x) = \lim_{x \to 1} 2x + 4 = 2 + 4 = 6$

$\Rightarrow \lim_{x \to 1} f(x) = 6$. Also $f(1) = 1^2 + 5 = 6$. Thus $\lim_{x \to 1} f(x) = f(1)$. Thus f is continuous at $x = 1$ also.

Hence f is continuous everywhere.

Example-2: Find the value of k so that the function f defined as $f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \ne \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$ is continuous at $x=\frac{\pi}{2}$.

Solution: For $x\neq\frac{\pi}{2}, f(x)=\frac{k\cos x}{\pi-2x}$, which is continuous being the quotient of two continuous functions $k\cos x$ and $\pi-2x$.

$$\lim_{x\to\frac{\pi}{2}-} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)}=\lim_{h\to 0}\frac{k\sin h}{2h}=\frac{k}{2}$$

$$\lim_{x\to\frac{\pi}{2}+} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}=\lim_{h\to 0}\frac{-k\sin h}{-2h}=\frac{k}{2}$$

and $f\left(\frac{\pi}{2}\right)=3$

Now f will be continuous if it is continuous at $x=\frac{\pi}{2}$.

$$\Rightarrow \lim_{x\to\frac{\pi}{2}-} f(x)=\lim_{x\to\frac{\pi}{2}+} f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow\frac{k}{2}=3\Rightarrow k=6$$

Example-3: Prove that the function f defined as $f(x)=\begin{cases}\frac{\sin x}{x} & \text{if } x>0\\ x+1 & \text{if } x<0\end{cases}$ is continuous everywhere.

Solution: For $x<0, f(x)=\frac{\sin x}{x}$. As $\sin x$ and x are everywhere continuous functions therefore $\frac{\sin x}{x}$ is continuous for $x<0$, i.e. $f(x)$ is continuous for x<0.

For $x>0$, $f(x)=x+1$. As $x+1$ is a polynomial function therefore x+1 is continuous for every x and in particular for $x>0$. Thus $f(x)$ is continuous for every $x>0$.

Now let us discuss the continuity of f at $x=0$

$\lim_{x\to 0-} f(x)=\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to 0+} f(x)=\lim_{x\to 0}(x+1)=1$. Thus $\lim_{x\to 0} f(x)=1$. Also $f(0)=1$.

Thus $\lim_{x\to 0} f(x)=f(0)$. Therefore f is continuous at $x=0$.

Hence f is everywhere continuous function.

Example-4: Show that the function f defined as $f(x)=|x|+|x-1|$ is everywhere continuous.

Solution: Here $f(x)=|x|+|x-1|=\begin{cases}-x-(x-1)=-2x+1 & \text{for } x<0\\ x-(x-1)=1 & \text{for } 0\leq x\leq 1\\ x+(x-1)=2\text{x}-1 & \text{for } x>1\end{cases}$

For $x<0, f(x)=-2\text{x}+1$, which is continuous being a polynomial function.

For $0<x<1, f(x)=1$ which is continuous being a polynomial function.

For $x>1, f(x)=2x-1$ which is continuous being a polynomial function.

$\lim_{x\to 0-} f(x)=\lim_{x\to 0}(-2x+1)=1$ And $\lim_{x\to 0+} f(x)=\lim_{x\to 0}(1)=1$. Also $f(0)=1$.

Thus $\lim_{x\to 0-} f(x)=\lim_{x\to 0+} f(x)=f(0)$. Therefore f is continuous at $x=0$.

Now $\lim_{x\to 1-} f(x)=\lim_{x\to 1}(1)=1$ and $\lim_{x\to 1+} f(x)=\lim_{x\to 1}(2x-1)=1$. Also $f(1)=1$

Thus $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(1)$. Therefore f is continuous at $x=1$.
Hence f is continuous everywhere.

Exercise

1. Discuss the continuity of the function f defined as $f(x) = \begin{cases} x^2+3 & \text{for } x \le 1 \\ 2x+2 & \text{for } x > 1 \end{cases}$

2. Discuss the continuity of the function f defined as $f(x) = \begin{cases} \dfrac{x}{\sin x} & \text{for } x < 0 \\ x+1 & \text{for } x \ge 0 \end{cases}$

3. Discus the continuity of the function f , defined as $f(x) = \begin{cases} \dfrac{|x-2|}{x-2} & \text{for } x \ne 2 \\ 1 & \text{for } x = 2 \end{cases}$

4. Discuss the continuity of the function f , defined as $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

5. Discuss the continuity of the function f defined as $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$

6. Show that the function f defined as $f(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x .