Properties of Determinant

Determinant: To each square matrix A we can associate a expression or number (real or complex) known as its determinant denoted by $\det(A)$ or $|A|$

If $A=[a]$ then $|A|=a$.

Determinant of a square matrix of order 2:

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A|=ad-bc$

For example $\begin{vmatrix} 2 & 3 \\ -3 & 4 \end{vmatrix}=(2\times4)-(3\times-3)=8+9=17$

Determinant of a square matrix matrix of order 3:

Let $A=\begin{bmatrix} a & b & c \\ x & y & z \\ u & v & w \end{bmatrix}$ then we can find its determinant in six possible ways

Expansion Along First Row:

$$|A|= a\begin{vmatrix} y & z \\ v & w \end{vmatrix} - b\begin{vmatrix} x & z \\ u & w \end{vmatrix} + c\begin{vmatrix} x & y \\ u & v \end{vmatrix}$$

Expansion Along Second Row:

$$|A|=-x\begin{vmatrix} b & c \\ v & w \end{vmatrix}+y\begin{vmatrix} a & c \\ u & w \end{vmatrix}-z\begin{vmatrix} a & b \\ u & v \end{vmatrix}$$

Expansion Along Third Row:

$$|A|=u\begin{vmatrix} b & c \\ y & z \end{vmatrix}-v\begin{vmatrix} a & c \\ x & z \end{vmatrix}+w\begin{vmatrix} a & b \\ x & y \end{vmatrix}$$

Expansion along first column:

$$|A|=a\begin{vmatrix} y & z \\ v & w \end{vmatrix}-x\begin{vmatrix} b & c \\ v & w \end{vmatrix}+u\begin{vmatrix} b & c \\ y & z \end{vmatrix}$$

Expansion along second column:

$$|A|=-b\begin{vmatrix} x & z \\ u & w \end{vmatrix}+y\begin{vmatrix} a & c \\ u & w \end{vmatrix}-v\begin{vmatrix} a & c \\ x & z \end{vmatrix}$$

Expansion along third column:

$$|A|=c\begin{vmatrix} x & y \\ u & v \end{vmatrix}-z\begin{vmatrix} a & b \\ u & v \end{vmatrix}+w\begin{vmatrix} a & b \\ x & y \end{vmatrix}$$

Note: We expand the determinant along a row or column which has maximum number of zeros.

Example-1: Evaluate the determinant $\begin{vmatrix} 2 & -3 & 1 \\ 3 & -2 & 4 \\ 1 & 2 & 5 \end{vmatrix}$ by expanding it along first row.

Solution: $\begin{vmatrix} 2 & -3 & 1 \\ 3 & -2 & 4 \\ 1 & 2 & 5 \end{vmatrix}=(2)\begin{vmatrix} -2 & 4 \\ 2 & 5 \end{vmatrix}-(-3)\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix}+(1)\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}$

$$\Rightarrow \begin{vmatrix} 2 & -3 & 1 \\ 3 & -2 & 4 \\ 1 & 2 & 5 \end{vmatrix}=2(-10-8)+3(15-4)+(6+2)=-36+33+8=5$$

Example-2: Find the value of the determinant $\begin{vmatrix} 2 & 3 & -1 \\ -1 & 2 & -4 \\ 2 & -1 & 3 \end{vmatrix}$ by expanding it along the second row.

Solution:

$$\begin{vmatrix} 2 & 3 & -1 \\ -1 & 2 & -4 \\ 2 & -1 & 3 \end{vmatrix} = -(-1)\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix} - (-4)\begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = (9-1)+2(6+2)+4(-2-6)$$

$$\Rightarrow \begin{vmatrix} 2 & 3 & -1 \\ -1 & 2 & -4 \\ 2 & -1 & 3 \end{vmatrix} = 8+16-32 = -8$$

Example-3: Find the value of the determinant $\begin{vmatrix} 3 & 2 & -1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \end{vmatrix}$ by expanding along it third row.

Solution: $\begin{vmatrix} 3 & 2 & -1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \end{vmatrix} = 4\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix}$

$$\Rightarrow \begin{vmatrix} 3 & 2 & -1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \end{vmatrix} = 4(2+1)+2(3-2)+3(3+4) = 12+2+21 = 35$$

Example-4: Find the value of the determinant $\begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 3 \\ 1 & 2 & 1 \end{vmatrix}$ by expanding along first column.

Solution: $\begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 3 \\ 1 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix}$

$$\begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 3 \\ 1 & 2 & 1 \end{vmatrix} = 2(-2-6)-3(1-2)+(3+2) = -16+3+5 = -8$$

Example-5: Find the value of the determinant $\begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & -1 \\ 5 & 2 & 1 \end{vmatrix}$ by expanding it along second column.

Solution: $\begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & -1 \\ 5 & 2 & 1 \end{vmatrix} = -(-1)\begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$

$$\Rightarrow \begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & -1 \\ 5 & 2 & 1 \end{vmatrix} = (4+5)+2(2-15)-2(-2-12) = 9-26+28 = 11$$

Example-6: Find the value of the determinant $\begin{vmatrix} 3 & -1 & -1 \\ 2 & 2 & -1 \\ 2 & 3 & -1 \end{vmatrix}$ by expanding it along the third column.

Solution: $\begin{vmatrix} 3 & -1 & -1 \\ 2 & 2 & -1 \\ 2 & 3 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix}$

$$\begin{vmatrix} 3 & -1 & -1 \\ 2 & 2 & -1 \\ 2 & 3 & -1 \end{vmatrix} = -(6-4)+(9+2)-(6+2) = -2+11-8 = 1$$

Exercise-4.1

1. Find the value of the determinant $\begin{vmatrix} 3 & 1 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 2 \end{vmatrix}$ by expanding it along the first row. Also find its value by expanding it along first column.
2. Find the value of the determinant $\begin{vmatrix} 3 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix}$ by expanding it along the second row. Also find its value by expanding it along second column.
3. Find the value of the determinant $\begin{vmatrix} 3 & 4 & 0 \\ -1 & 2 & 3 \\ 2 & 4 & 5 \end{vmatrix}$ by expanding it along the third row. Also find its value by expanding it along third column.
4. Find the value of the determinant $\begin{vmatrix} a & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix}$
5. Find the value of the determinant $\begin{vmatrix} 2 & 0 & 1 \\ -1 & 0 & 2 \\ 1 & 2 & 1 \end{vmatrix}$
6. Find the value of the determinant $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$
7. Find the values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$
8. Find the value of thew determinant $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$
9. Find the value of the determinant $\begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$

Properties of Determinant

Property-1: Let $A=[a_{ij}]$ be a square matrix of order n then $|A|=|A^T|$

For example $\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & x & u \\ b & y & v \\ c & z & w \end{vmatrix}$

Property-2: Let $A=[a_{ij}]$ be a square matrix of order $n\,(n\geq 2)$ and B be matrix obtained from A by interchanging any two rows(columns) of A, then $|A|=-|B|$

For example $\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} = -\begin{vmatrix} x & y & z \\ a & b & c \\ u & v & w \end{vmatrix}$ or $\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} = -\begin{vmatrix} b & a & c \\ y & x & z \\ v & u & w \end{vmatrix}$

Property-3: If any two rows(columns) of a square matrix $A=[a_{ij}]$ of order n are identical, then its determinant is zero i.e. $|A|=0$

For example $\begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix} = 0$ or $\begin{vmatrix} a & b & a \\ x & y & x \\ u & v & u \end{vmatrix} = 0$

Property-4: Let $A=[a_{ij}]$ be a square matrix of order n, and let B be the matrix obtained from A by multiplying each element of a row(column) of A by a scalar λ then $|B|=k|A|$

For example $\begin{vmatrix} ka & kb & kc \\ x & y & z \\ u & v & w \end{vmatrix} = k\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}$ or $\begin{vmatrix} ka & b & c \\ kx & y & z \\ ku & v & w \end{vmatrix} = k\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}$

Property-5: Let A be a square matrix such that each element of a row(column) of A is expressed as the sum of two or more terms. Then, the determinant of A can be expressed as the sum of determinants of two or more matrices of of the same order.

For example $\begin{vmatrix} a+a' & b+b' & c+c' \\ x & y & z \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} + \begin{vmatrix} a' & b' & c' \\ x & y & z \\ u & v & w \end{vmatrix}$

or $\begin{vmatrix} a & b & c \\ x+x' & y & z \\ u+u' & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} + \begin{vmatrix} a' & b & c \\ x' & y & z \\ u' & v & w \end{vmatrix}$

Property-6: Let A be any square matrix and B be a matrix obtained from A by adding to a row(column) of A a scalar multiple of another row(column) of A, then $|B|=|A|$

For example $\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} = \begin{vmatrix} a+ku & b+kv & c+kw \\ x & y & z \\ u & v & w \end{vmatrix}$ or

$\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} = \begin{vmatrix} a+kc & b & c \\ x+kz & y & z \\ u+kw & v & w \end{vmatrix}$

Example1. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}=0$

Solution: Let $\Delta=\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$

Applying $R_2 \to R_2 + R_3$ we get

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ a+b+c & a+b+c & a+b+c \\ b+c & c+a & a+b \end{vmatrix}$$

Taking $(a+b+c)$ common from R_2 we get

$$\Delta=(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ b+c & c+a & a+b \end{vmatrix}$$

$\Rightarrow \Delta=0$ $\quad (\because R_1 \text{ and } R_2 \text{ are identical})$

Example-2: Prove that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}=0$

Solution: Let $\Delta=\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

Applying $R_1 \to R_1 + C_2 + R_3$

$$\Delta=\begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$\Rightarrow \Delta=0$ $\quad$ ($\because$ all elements in R_1 are zero)

Example-3 Prove that $\begin{vmatrix} 1 & 1 & 1 \\ ab & bc & ca \\ c(a+b) & a(b+c) & b(c+a) \end{vmatrix}=0$

Solution: Let $\Delta=\begin{vmatrix} 1 & 1 & 1 \\ ab & bc & ca \\ c(a+b) & a(b+c) & b(c+a) \end{vmatrix}$

Applying $R_2 \to R_2 + R_3$ we get

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ ab+bc+ca & ab+bc+ca & ab+bc+ca \\ c(a+b) & a(b+c) & b(c+a) \end{vmatrix}$$

$$\Rightarrow \Delta=(ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ c(a+b) & a(b+c) & b(c+a) \end{vmatrix}$$

$\Rightarrow \Delta=0$ $\quad$ (As R_1 and R_2 are identical)

Example-4: Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution: Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ we get

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

Taking $(a-b)$ and $(b-c)$ common from C_1 and C_2 respectively we get

$$\Delta = (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

Expanding along R_1 we get

$$\Delta = (a-b)(b-c)\{(b+c)-(a+b)\} = (a-b)(b-c)(c-a)$$

Example-5 Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Solution: Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ we get

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$\Rightarrow \Delta = (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ we get

$$\Delta = (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ a^2+ab-bc-c^2 & b^2+bc+c^2 & c^2 \end{vmatrix}$$

$$\Delta = (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ (a-c)(a+b+c) & b^2+bc+c^2 & c^2 \end{vmatrix}$$

$$\Delta = (a-b)(b-c)(a-c)(a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ 1 & b^2+bc+c^2 & c^2 \end{vmatrix}$$ (Taking out $(a-c)(a+b+c)$ from C_1)

$$\Delta=(a-b)(b-c)(a-c)(a+b+c)1\cdot\begin{vmatrix}0 & 1\\ 1 & b^2+bc+c^2\end{vmatrix}$$ (Expanding along R_3)

$$\Delta=(a-b)(b-c)(a-c)(a+b+c)\times -1$$

$$\Delta=(a-b)(b-c)(c-a)(a+b+c)$$

Example 6: Prove that $\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}=(a+b+c)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=abc+bc+ca+ab$

Solution: Let $\Delta=\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}$

Taking a, b, c common from $R_{1,} R_2$ and R_3 we get

$$\Delta=abc\begin{vmatrix}\frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a}\\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b}\\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1\end{vmatrix}$$

Applying $R_1 \rightarrow R_2+R_3$ we get

$$\Delta=abc\begin{vmatrix}1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b}\\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c}\end{vmatrix}$$

Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from R_1 we get

$$\Delta=abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix}1 & 1 & 1\\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b}\\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c}\end{vmatrix}$$

Applying $C_1 \rightarrow C_1-C_2$ and $C_2 \rightarrow C_2-C_3$ we get

$$\Delta=abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix}0 & 0 & 1\\ -1 & 1 & \frac{1}{b}\\ 0 & -1 & 1+\frac{1}{c}\end{vmatrix}$$

Expanding along R_1 we get

$$\Delta=abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)1\cdot\begin{vmatrix}-1 & 1\\ 0 & -1\end{vmatrix}=abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=abc+bc+ca+ab$$

Example-7 Prove that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

Solution: Let $\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

Applying $C_1 \to C_1 - bC_3$ and $C_2 \to C_2 + aC_3$ we get

$$\Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

Taking $(1+a^2+b^2)$ common from both C_1 and C_2 we get

$$\Delta = (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

Applying $R_3 \to R_3 - bR_1 + aR_2$ we get

$$\Delta = (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix}$$

Expanding along R_3 we get

$$\Delta = (1+a^2+b^2)^2 (1+a^2+b^2) \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (1+a^2+b^2)^3$$

Example 8: Prove $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

Solution: Let $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$ we get

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

Taking $(a+b+c)$ common from C_1 and C_3 we get

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

Applying $R_3 \to R_3 - (R_1 + R_2)$ we get

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \frac{(a+b+c)^2}{ab}\begin{vmatrix} ab+ac-a^2 & 0 & a^2 \\ 0 & bc+ab-b^2 & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix} \quad (C_1 \to aC_1 \text{ and } C_2 \to bC_2)$$

Applying $C_1 \to C_1 + C_3$ and $C_2 \to C_2 + C_3$

$$\Delta = \frac{(a+b+c)^2}{(ab)}\begin{vmatrix} ab+ac & a^2 & a^2 \\ b^2 & bc+ab & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

$$\Delta = \frac{(a+b+c)^2}{ab} 2ab \cdot \begin{vmatrix} ab+bc & a^2 \\ b^2 & bc+ab \end{vmatrix} \qquad \text{(By Expanding along } R_3\text{)}$$

Taking a common from R_1 and b common from R_2 we get

$$\Delta = 2ab(a+b+c)^2 \begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix}$$

$$\Rightarrow \quad \Delta = 2\text{ab}(a+b+c)^2 \{bc+ba+c^2+ac-ab\}$$

$$\Rightarrow \Delta = 2\text{abc}(a+b+c)^3$$

Example 9: Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution: Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ we get

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ -c(a-b) & -a(b-c) & ab \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ -c & -a & ab \end{vmatrix} \qquad \text{(Taking } (a-b) \text{ common from } C_1 \text{ and } (b-c) \text{ from } C_2\text{)}$$

Expanding along R_1 we get

$$\Delta = (a-b)(b-c)(c-a)1 \cdot \begin{vmatrix} 1 & 1 \\ -c & -a \end{vmatrix}$$

$$\Rightarrow \Delta = (a-b)(b-c)(c-a)$$

Example-10 Prove that $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 4xyz$

Solution: Let $\Delta = \begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$ we get

$$\Delta = \begin{vmatrix} y+z-x & 0 & x \\ 0 & z+x-y & y \\ z-x-y & z-x-y & x+y \end{vmatrix}$$

Applying $R_3 \to R_3 - (R_1 + R_2)$ we get

$$\Delta = \begin{vmatrix} y+z-x & 0 & x \\ 0 & z+x-y & y \\ -2y & -2x & 0 \end{vmatrix}$$

Expanding along R_1 we get

$$\Delta = (y+z-x)(2xy) + x(2y)(z+x-y)$$

$$\Rightarrow \Delta = 2xy(y+z-x+z+x-y) = 4xyz$$

Example-11 Prove that $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

Solution: Let $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

Applying $C_1 \to C_2 + C_3$ we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(p+q+r) & r+p & p+q \\ 2(x+y+z) & z+x & x+y \end{vmatrix}$$

$$\Rightarrow \Delta = 2\begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ we get

$$\Delta = 2\begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$ we get

$$\Delta = 2\begin{vmatrix} a & -b & -c \\ p & -q & -r \\ x & -y & -z \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

Example-12: Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$

Solution: Let $\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$

$$\Rightarrow \Delta = \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2 b & a^2 c \\ ab^2 & b(b^2+1) & b^2 c \\ c^2 a & c^2 b & c(c^2+1) \end{vmatrix} \quad \left(R_1 \to \frac{1}{a}R_1,\ R_2 \to \frac{1}{b}R_2 \text{ and } R_3 \to \frac{1}{c}R_3 \right)$$

$$\Rightarrow \Delta = \frac{abc}{abc}\begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$ we get

$$\Delta = \begin{vmatrix} 1+a^2+b^2 & 1+a^2+b^2 & 1+a^2+b^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

$$\Delta = (1+a^2+b^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$ (By taking $1+a^2+b^2$ coomon from R_1)

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ we get

$$\Delta = (1+a^2+b^2)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & b^2 \\ 0 & -1 & c^2+1 \end{vmatrix}$$

Expanding along R_1 we get

$$\Delta = (1+a^2+b^2)\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = (1+a^2+b^2)$$

Example-13: Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Solution: Let $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$ we get

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ we get

$$\Delta = (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -(a+b+c) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$$

$$\Delta = (a+b+c)^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$ (Taking $a+b+c$ common from C_1 and C_2)

$$\Delta=(a+b+c)^3\begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix}$$ (Expandibg along R_1)

$$\Rightarrow\Delta=(a+b+c)^3$$

Example 14: Prove that $\begin{vmatrix}\sin\alpha & \cos\alpha & \cos(\alpha+\delta)\\ \sin\beta & \cos\beta & \cos(\beta+\delta)\\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta)\end{vmatrix}=0$

Solution: Let $\Delta=\begin{vmatrix}\sin\alpha & \cos\alpha & \cos(\alpha+\delta)\\ \sin\beta & \cos\beta & \cos(\beta+\gamma)\\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta)\end{vmatrix}$

$$\Rightarrow\Delta=\begin{vmatrix}\sin\alpha & \cos\alpha & \cos\alpha\cos\delta-\sin\alpha\sin\delta\\ \sin\beta & \cos\beta & \cos\beta\cos\delta-\sin\beta\sin\delta\\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta-\sin\gamma\sin\delta\end{vmatrix}$$

By applying $C_3\rightarrow C_3+\sin\delta\, C_1-\cos\delta\, C_2$

$$\Rightarrow\Delta=\begin{vmatrix}\sin\alpha & \cos\alpha & 0\\ \sin\beta & \cos\beta & 0\\ \sin\gamma & \cos\gamma & 0\end{vmatrix}$$

$$\Rightarrow\Delta=0$$

Example-15: Prove that $\begin{vmatrix}a^2 & bc & ac+c^2\\ a^2+ab & b^2 & ac\\ ab & b^2+bc & c^2\end{vmatrix}=4a^2b^2c^2$

Solution: Let $\Delta=\begin{vmatrix}a^2 & bc & ac+c^2\\ a^2+ab & b^2 & ac\\ ab & b^2+bc & c^2\end{vmatrix}$

$$\Rightarrow\Delta=a\,b\,c\begin{vmatrix}a & c & a+c\\ a+b & b & a\\ b & b+c & c\end{vmatrix}$$

Applying $R_1\rightarrow R_1+R_2+R_3$ we get

$$\Delta=abc\begin{vmatrix}2(a+b) & 2(b+c) & 2(c+a)\\ a+b & b & a\\ b & b+c & c\end{vmatrix}$$

$$\Delta=2abc\begin{vmatrix}a+b & b+c & c+a\\ a+b & b & a\\ b & b+c & c\end{vmatrix}$$

Applying $R_1\rightarrow R_1-R_2$ and we get

$$\Delta=2abc\begin{vmatrix}0 & c & c\\ a+b & b & a\\ b & b+c & c\end{vmatrix}$$

Applying $C_2\rightarrow C_2-C_3$ we get

$$\Delta = 2abc\begin{vmatrix} 0 & 0 & c \\ a+b & b-a & a \\ b & b & c \end{vmatrix}$$

$$\Delta = 2abc\{c(ab+b^2-b^2+ab)\} = 4a^2b^2c^2$$

Example-16 If a, b, c are positive and unequal, show that the value of the determinant

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

is negative.

Solution: Applying $C_1 \to C_1 + C_2 + C_3$ we get

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix} \quad (\text{Applying } R_1 \to R_1 - R_2 \text{ and } R_2 \to R_2 - R_3)$$

$$\Rightarrow \Delta = (a+b+c)\{(b-c)(a-b)-(c-a)^2\}$$

$$\Rightarrow \Delta = (a+b+c)(ba-b^2-ca+bc-c^2-a^2+2ca)$$

$$\Rightarrow \Delta = (a+b+c)(-a^2-b^2-c^2+ab+bc+ca) = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$\Rightarrow \Delta = \frac{-1}{2}(a+b+c)(2a^2+2b^2+2c^2-2ab-2bc-2ca)$$

$$\Rightarrow \Delta = \frac{-1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

As a, b, c are positive and unequal therefore $a+b+c > 0$ and $(a-b)^2+(b-c)^2+(c-a)^2 > 0$

$\Rightarrow \Delta < 0$

Example 17: Solve $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

Solution: Let $\Delta = \begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$ we get $\Delta = \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix}$

$$\Rightarrow \Delta = (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get

$$\Delta = (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix}$$

$$\Rightarrow \Delta = (3a-x)\times 1\times\begin{vmatrix} 2x & 0 \\ 0 & 2x \end{vmatrix} \quad \text{(Expanding along } C_1\text{)}$$

$$\Rightarrow \Delta = (3a-x)4x^2$$

Thus $\Delta = 0 \Rightarrow (3a-x)4x^2 = 0 \Rightarrow x = 0, 3a$

Example 18: Prove that $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

Solution: We have $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

$$\Rightarrow \Delta = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad \{\text{As each element of } C_3 \text{ is sum of two elements}\}$$

$$\Rightarrow \Delta = -\begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\Rightarrow \Delta = (1+pxyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\Rightarrow \Delta = (1+pxyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \quad \text{(Applying } R_2 \to R_2 - R_3 \text{ and } R_3 \to R_3 - R_1\text{)}$$

$$\Rightarrow \Delta = (1+pxyz)(y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

$$\Rightarrow \Delta = (1+pxyz)(y-x)(z-x)\begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix} \quad \text{(Expanding along } C_1\text{)}$$

$$\Rightarrow \Delta = (1+pxyz)(y-z)(z-x)(z+x-y-x) = (1+pxyz)(x-y)(y-z)(z-x)$$

Example-19: Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Solution: Let $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Applying $C_1 \to \frac{1}{a}C_1$, $C_2 \to \frac{1}{b}C_2$ and $C_3 \to \frac{1}{c}C_3$ we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \Leftrightarrow R_3$ we get

$$\Delta = -\begin{vmatrix} 1 & 1 & 1 \\ a^3 & b^3 & c^3 \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Applying $R_2 \Leftrightarrow R_3$ we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ we get

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a^2 - b^2 & b^2 - c^2 & c^2 \\ a^3 - b^3 & b^3 - c^3 & c^3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ (a-b)(a+b) & (b-c)(b+c) & c^2 \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$\Rightarrow \Delta = (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ a+b & b+c & c^2 \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ we get

$$\Delta = (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ a+b & c-a & c^2 \\ a^2+ab+b^2 & (c-a)(a+b+c) & c^3 \end{vmatrix}$$

$$\Rightarrow \Delta = (a-b)(b-c)(c-a)\begin{vmatrix} 0 & 0 & 1 \\ a+b & 1 & c^2 \\ a^2+ab+b^2 & a+b+c & c^3 \end{vmatrix}$$

Expanding along R_1 we get

$\Delta = (a-b)(b-c)(c-a)\{(a+b)(a+b+c) - (a^2+ab+b^2)\}$

$\Rightarrow \Delta = (a-b)(b-c)(c-a)(a^2+ab+ac+ab+b^2+bc-a^2-ab-b^2)$

$\Rightarrow \Delta = (a-b)(b-c)(c-a)(ab+bc+ca)$

Example-20: Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

Solution: Let $\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$

Since each element of the second column is sum of two elements we get

$$\Delta = \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + \begin{vmatrix} a & b & a+b+c \\ 2a & 2b & 4a+3b+2c \\ 3a & 3b & 10a+6b+3c \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + ab \cdot \begin{vmatrix} 1 & 1 & a+b+c \\ 2 & 2 & 4a+3b+2c \\ 3 & 3 & 10a+6b+3c \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + ab \cdot 0$$ [$\because$ C_2 and C_3 are identical in second determinant)

As each element of C_3 is sum of three elements therefore

$$\Delta = \begin{vmatrix} a & a & a \\ 2a & 3a & 4a \\ 3a & 6a & 10a \end{vmatrix} + \begin{vmatrix} a & a & b \\ 2a & 3a & 3b \\ 3a & 6a & 6b \end{vmatrix} + \begin{vmatrix} a & a & c \\ 2a & 3a & 2c \\ 3a & 6a & 3c \end{vmatrix}$$

$$\Rightarrow \Delta = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + a^2 b \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 6 & 6 \end{vmatrix} + a^2 c \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 6 & 3 \end{vmatrix}$$

As C_2 and C_3 are identical in second determinant and C_1 and C_3 are identical in third determinant thus we get

$$\Delta = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + a^2\ b \cdot 0 + a^2\ c \cdot 0$$

$$\Rightarrow \Delta = a^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix}$$ [Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$]

$\Rightarrow \Delta = a^3(7-6)$ [Expanding along R_1]

$\Rightarrow \Delta = a^3$

Exercise 4.2

Using Properties of determinants, prove the following

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

2. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

3. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

4. $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 0$

5. $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$

6. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

7. $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & 2x+4 \end{vmatrix} = (5x+4)(4-x)^2$

8. $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3\mathrm{y}+k)$

9. $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$

10. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

11. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

12. $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

13. $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^2)^3$

14. $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$

15. $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc-a^3-b^3-c^3$

16. $\begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$

17. $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

18. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

19. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$

20. Prove that $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ

21. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

22. Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$, $a \neq 0$

23. If $x \neq y \neq z$ and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then prove that $xyz = -1$

24. Solve $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

25. If a, b, c are in A.P, find value of $\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$

Area of Triangle

Result : Area of triangle whose vertices are $(x_1, y_1, z_1), (x_2, y_2, z_2); (x_3, y_3, z_3)$ is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Collinearity of three points: Three points $(x_1, y_1, z_1), (x_2, y_2, z_2); (x_3, y_3, z_3)$ will be collinear if and only if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Example-1: Find the area of the triangle whose vertices are $(1,2), (2;3), (3;-1)$

Solution: $\Delta = \pm\frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & -1 & 1 \end{vmatrix} = \pm\frac{1}{2}\{1(3+1) - 2(2-3) + 1(-2-9)\}$

$\Rightarrow \Delta = \pm\frac{1}{2}(4+2-11) = \frac{5}{2}$ square units

Example-2: Show that the points $(1,2); (4,-1); (5.-2)$ are collinear.

Solution: $\begin{vmatrix} 1 & 2 & 1 \\ 4 & -1 & 1 \\ 5 & -2 & 1 \end{vmatrix} = 1(-1+2)-2(4-5)+1(-8+5)=1+2-3=0$

Example-3: If points $(a,0),(0,b),(x,y)$ are collinear then show that $\frac{x}{a}+\frac{y}{b}=1$

Solution: As points $(a,0),(0,b),(x,y)$ are collinear $\begin{vmatrix} a & 0 & 1 \\ 0 & b & 1 \\ x & y & 1 \end{vmatrix}=0$

$\Rightarrow a(b-y)-bx=0 \quad \Rightarrow ab-ay-bx=0 \Rightarrow ay+bx=ab$

Divide both sides by ab we get

$\frac{x}{a}+\frac{y}{b}=1$

Exercise 4.3

1. Find the area of the triangle whose vertices are
   (i) $(2,3),(4,-1),(0,2)$ (ii) $(3,-1),(2,4),(9,2)$ (iii) $(4,-1),(4,1),(7,3)$
2. Show that the points $(3,1),(4,0),(6,-2)$ are collinear.
3. Show that the points $(3,1),(2,-1),(1,-3)$ are collinear.
4. If the points $(a,0),(0,b),(1,1)$ are collinear then show $a+b=ab$
5. Using determinants prove that the points $(a,b),(a',b')$ and $(a-a',b-b')$ are collinear if $ab'=ba'$
6. Find the value of λ such that $(2,3),(\lambda,1),(-1,2)$ are collinear.
7. Using determinants find the equation of the line joining the points $(2,3)$ and $(-3,1)$

Minors and Co-Factors

Minors: Let $A=[a_{ij}]$ be a square matrix of order n. Then we define minor of a_{ij} as a determinant of order $n-1$ obtained from A be deleting its i^{th} row and j^{th} column. Minor of a_{ij} is denoted by M_{ij}

Co-Factor: Let $A=[a_{ij}]$ be a square matrix of order n. Then we define co-factor of a_{ij} as $(-)^{i+j}M_{ij}$. We denote the co-factor of a_{ij} as A_{ij} or C_{ij}

Thus $A_{ij}=(-1)^{i+j}M_{ij}$

or $A_{ij}=\begin{cases} M_{ij} \text{ if } i+j \text{ is } even \\ -M_{ij} \text{ if } i+j \text{ is } odd \end{cases}$

Result: If $A=[a_{ij}]$ be a square matrix of order n

$|A|=\sum_{j=1}^{n} a_{ij}A_{ij}$ (Expansion along i^{th} row)

$|A|=\sum_{j=1}^{n} a_{ij}C_{ij}$ (Expanding along j^{th} column)

To be more specific let $A=[a_{ij}]$ be a square matrix of order 3.

Then $|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$ (Expansion along first column)

$|A|=a_{21}A_{21}+a_{22}A_{22}+a_{23}A_{23}$ (Expanding along second row)

$|A|=a_{31}A_{31}+a_{32}A_{32}+a_{33}A_{33}$ (Expanding along third row)

$|A|=a_{11}A_{11}+a_{21}A_{21}+a_{31}C_{31}$ (Expansion along first column)

$|A|=a_{12}A_{12}+a_{22}A_{22}+a_{32}A_{32}$ (Expanding along second column)

$|A|=a_{13}A_{13}+a_{23}A_{23}+a_{33}A_{33}$ (Expanding along third column)

Note: If we multiply the elements of a row(column) by the co-factors of elements of different row(column) and then add the value of such expression is zero.

For example for a square matrix of order 3
$a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{23}=0$

Example-1: Find the minor and co-factor of each element of the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 2 \\ 2 & 5 & 1 \end{bmatrix}$

Solution: $M_{11}=\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix}=4-10=-6$; $A_{11}=-6$

$M_{12}=\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}=-1-4=-5, A_{12}=5$

$M_{13}=\begin{vmatrix} -1 & 4 \\ 2 & 5 \end{vmatrix}=-5-8=-13, A_{13}=-13$

$M_{21}=\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix}=3-5=-2, A_{21}=2$

$M_{22}=\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}=2-2=0, A_{22}=0$

$M_{23}=\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}=10-9=1, A_{23}=-1$

$M_{31}=\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}=6-4=2, A_{31}=2$

$M_{32}=\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}=4+1=5, A_{32}=-5$

$M_{33}=\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix}=8+3=11, A_{33}=11$

Example-2: Find the determinant of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 3 & 2 & 5 \end{bmatrix}$ and verify the relation

$|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$

Solution: $|A|=2\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix}-3\begin{vmatrix} -1 & 3 \\ 3 & 5 \end{vmatrix}+4\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix}=2(10-6)-3(-5-9)+4(-2-6)=8+42-32=18$

$A_{11}=(-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix}=10-6=4$, $A_{12}=(-1)^{1+2}\begin{vmatrix} -1 & 3 \\ 3 & 5 \end{vmatrix}=-(-5-9)=14$

$A_{13}=(-1)^{1+3}\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix}=-2-6=-8$

Thus $a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}=2(4)+3(14)+4(-8)=8+42-32=18$

Thus $|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$

Example-3: For the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$, verify that $a_{21}A_{11}+a_{22}A_{12}+a_{23}A_{13}=0$

Solution: $A_{11}=(-1)^{1+1}\begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix}=2-8=-6$

$A_{12}=(-1)^{1+2}\begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix}=-(1+4)=-5$

$A_{13}=(-1)^{1+3}\begin{vmatrix}1 & 2\\ -1 & 2\end{vmatrix}=2+2=4$

Thus $a_{21}A_{11}+a_{22}A_{12}+a_{23}A_{13}=1(-6)+2(-5)+4(4)=-6-10+16=0$

Exercise-4.4

1. Write the minors and co-factors of each element the determinant $\begin{vmatrix}2 & 3\\ 1 & 4\end{vmatrix}$
2. Write the minors and cofactors of each element of the determinant $\begin{vmatrix}2 & -1 & 3\\ -2 & 4 & -1\\ 5 & 1 & 2\end{vmatrix}$
3. Using co-factors of the elements of the 2nd column find the value fo the determinant $\begin{vmatrix}2 & -1 & 0\\ 3 & 2 & -1\\ 2 & 1 & 4\end{vmatrix}$
4. Using co-factors of elements of 3rd row find the value of the determinant $\begin{vmatrix}3 & -1 & 1\\ -1 & 0 & 2\\ 3 & 5 & 1\end{vmatrix}$
5. For the determinant $\begin{vmatrix}2 & -1 & 3\\ 4 & 1 & 5\\ 1 & 0 & 2\end{vmatrix}$ show that $a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{23}=0$
6. For the determinant $\begin{vmatrix}-1 & 2 & 5\\ 1 & 2 & -1\\ 2 & 5 & 7\end{vmatrix}$, show that $a_{13}A_{12}+a_{23}A_{22}+a_{33}A_{23}=0$

Adjoint of a Matrix

Definition: : The adjoint of a square matrix $A=[a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$, where A_{ij} is the co-factor of the element a_{ij}. Adjoint of the matrix A is denoted by $\text{adj}(A)$

Example-1: Find the adjoint of the matrix $\begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix}$

Solution: $A_{11}=1, AA_{12}=-4, A_{21}=-3, A_{22}=2$

Hence $\text{adj}(A)=\text{transpose of}\begin{bmatrix}1 & -4\\ -3 & 2\end{bmatrix}=\begin{bmatrix}1 & -3\\ -4 & 2\end{bmatrix}$

Note: For a 2×2 matrix $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ $\text{adj}(A)=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

Thus the rule to write the adjoint of 2×2 matrix is that
(i) interchange the position of diagonal elements and
(ii) change the sign of the non-diagonal elements

Result: $|\text{adj}(A)|=|A|^{n-1}$

Result: $A\,\text{adj}(A)=|A|I$

Example-2: Find the adjoint of the matrix $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 1 \\ 2 & 5 & 1 \end{bmatrix}$

Solution: $A_{11}=-1, A_{12}=3, A_{13}=-13$; $A_{21}=14, A_{22}=-4, A_{23}=-8$; $A_{31}=-11, A_{32}=-5, A_{33}=9$

$$\text{adj}(A)=\begin{bmatrix} -1 & 14 & -11 \\ 3 & -4 & -5 \\ -13 & -8 & 9 \end{bmatrix}$$

Example 3: Find the adjoint of the matrix $A=\begin{bmatrix} 2 & 3 & -1 \\ 2 & 1 & -1 \\ 4 & 1 & 2 \end{bmatrix}$ and verify that $A\,\text{adj}(\text{A})=|A|I$. Also verify that $|\text{adj}(A)|=|A|^{n-1}$

Solution: $A_{11}=3, A_{12}=-8, A_{13}=-2; A_{21}=-7, A_{22}=8, A_{23}=10; A_{31}=-2, A_{32}=0, A_{33}=-4$

Thus $\text{adj}(A)=\begin{bmatrix} 3 & -7 & -2 \\ -8 & 8 & 0 \\ -2 & 10 & -4 \end{bmatrix}$

and $|A|=2(3)+3(-8)-(-2)=6-24+2=-16$

Now $A\,\text{adj}(A)=\begin{bmatrix} 2 & 3 & -1 \\ 2 & 1 & -1 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -7 & -2 \\ -8 & 8 & 0 \\ -2 & 10 & -4 \end{bmatrix}=\begin{bmatrix} -16 & 0 & 0 \\ 0 & -16 & 0 \\ 0 & 0 & -16 \end{bmatrix}=-16I=|A|I$

Now

$$|\text{adj}(A)|=\begin{vmatrix} 3 & -7 & -2 \\ -8 & 8 & 0 \\ -2 & 10 & -4 \end{vmatrix}=3(-32-0)-(-7)(32-0)-2(-80+16)=-96+224+128=256=(-16)^2$$

Thus $|\text{adj}(\text{A})|=|A|^2=|A|^{3-1}=|A|^{n-1}$ (Here $n=3$)

Exercise 4.5

1. Find the adjoint of the matrix $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ and verify the result $A\,\text{adj}(A)=|A|I$
2. Find the adjoint of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -3 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$ and verify the result $A\,\text{adj}(A)=|A|I$
3. Find the adjoint of the matrix $\begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix}$ and verify the result $|\text{adj}(A)|=|A|^{n-1}$
4. Find the adjoint of the matrix $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}$ and verify the result $|\text{adj}(A)|=|A|^{n-1}$
5. For the matrix $A=\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 3 & 5 & 5 \end{bmatrix}$ show that $A\,\text{adj}(A)=0$

Inverse of a Matrix

Singular Matrix: A square matrix A is said to be singular if $|A|=0$
Non-Singular Matrix: A square matrix A is said to be non-singular if $|A|\neq 0$
Result: If A and B are nonsingular matrices of the same order, then AB and BA are also non

singular matrices of the same order.

Result: $|AB|=|A||B|$ where A and B are square matrices of the same order.

Invertible Matrix: A square matrix A is invertible if and only if A is nonsingular.

and $$\boxed{A^{-1}=\frac{1}{|A|}\text{adj}(A)}$$

Example-1 Show that the matrix $\begin{bmatrix}2 & 3\\1 & 3\end{bmatrix}$ is invertible and find its inverse.

Solution: Let $A=\begin{bmatrix}2 & 3\\1 & 3\end{bmatrix}$. Then

$|A|=6-3=3\neq 0 \quad \Rightarrow A$ is nonsingular. Thus A is invertible.

$$A_{11}=3, A_{12}=-1, A_{21}=-3, A_{22}=2 \quad \Rightarrow \text{adj}(A)=\begin{bmatrix}3 & -3\\-1 & 2\end{bmatrix}$$

Thus $$A^{-1}=\frac{1}{|A|}\text{adj}(A)=\frac{1}{3}\begin{bmatrix}3 & -3\\-1 & 2\end{bmatrix}=\begin{bmatrix}1 & -1\\-\frac{1}{3} & \frac{2}{3}\end{bmatrix}$$

Example-2: Show that $A=\begin{bmatrix}2 & 3 & -1\\1 & 4 & 2\\2 & -1 & 3\end{bmatrix}$ is invertible and find A^{-1}

Solution: $$|A|=\begin{vmatrix}2 & 3 & -1\\1 & 4 & 2\\2 & -1 & 3\end{vmatrix}=2(12+2)-3(3-4)+(-1)(-1-8)=28+3+9=40$$

$$A_{11}=14, A_{12}=1, A_{13}=-9, A_{21}=-8, A_{22}=8, A_{23}=8, A_{31}=10, A_{32}=-5, A_{33}=5$$

Thus $$\text{adj}(A)=\begin{bmatrix}14 & -8 & 10\\1 & 8 & -5\\-9 & 8 & 5\end{bmatrix}$$

$$\therefore \quad A^{-1}=\frac{1}{|A|}\text{adj}(A)$$

$$\Rightarrow A^{-1}=\frac{1}{40}\begin{bmatrix}14 & -8 & 10\\1 & 8 & -5\\-9 & 8 & 5\end{bmatrix}=\begin{bmatrix}\frac{7}{20} & -\frac{1}{5} & \frac{1}{4}\\\frac{1}{40} & \frac{1}{5} & \frac{-1}{8}\\-\frac{9}{40} & \frac{1}{5} & \frac{1}{8}\end{bmatrix}$$

Example-3:: If $A=\begin{bmatrix}2 & 3\\1 & -4\end{bmatrix}$ and $B=\begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}$, then verify that $(AB)^{-1}=B^{-1}A^{-1}$

Solution: We have $$AB=\begin{bmatrix}2 & 3\\1 & -4\end{bmatrix}\begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}=\begin{bmatrix}-1 & 5\\5 & -14\end{bmatrix}$$

Since $|AB|=14-25=-11\neq 0$, $(AB)^{-1}$ exists and is given by

$$(AB)^{-1}=\frac{1}{|AB|}\text{adj}(AB)=-\frac{1}{11}\begin{bmatrix}-14 & -5\\-5 & -1\end{bmatrix}=\frac{1}{11}\begin{bmatrix}14 & 5\\5 & 1\end{bmatrix}$$

Further $|A|=-11\neq 0$ and $|B|=1\neq 0$. Therefore, A^{-1} and B^{-1} both exist and are given by

$$A^{-1}=-\frac{1}{11}\begin{bmatrix}-4 & -3\\-1 & 2\end{bmatrix}, B^{-1}=\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$$

Therefore $B^{-1}A^{-1} = -\frac{1}{11}\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}-4 & -3\\-1 & 2\end{bmatrix} = \frac{1}{11}\begin{bmatrix}14 & 5\\5 & 1\end{bmatrix}$

Hence $(AB)^{-1} = B^{-1}A^{-1}$

Example-4: If $A = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$ then show that $A^2 - 7A - 2I = 0$ and hence find A^{-1}.

Solution: $A^2 = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix} = \begin{bmatrix}16 & 21\\28 & 37\end{bmatrix}$, $7\text{A} = 7\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix} = \begin{bmatrix}14 & 21\\28 & 35\end{bmatrix}$, $2\text{I} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$

Thus $A^2 - 7\text{A} - 2\text{I} = \begin{bmatrix}16 & 21\\28 & 37\end{bmatrix} - \begin{bmatrix}14 & 21\\28 & 35\end{bmatrix} - \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

Thus $A^2 - 7A - 2I = 0$

Pre-Multiply by A^{-1} we get

$A^{-1}A^2 - 7A^{-1}A - 2A^{-1}A = 0 \Rightarrow A - 7I - 2A^{-1} = 0$

$\Rightarrow 2\text{A}^{-1} = A - 7\text{I} = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix} - \begin{bmatrix}7 & 0\\0 & 7\end{bmatrix} = \begin{bmatrix}-5 & 3\\4 & -2\end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix}-5 & 3\\4 & -2\end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix}-\frac{5}{2} & \frac{3}{2}\\2 & -1\end{bmatrix}$

Example-5: If $A = \begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{bmatrix}$ and $B = \begin{bmatrix}3 & -1 & -1\\-1 & 3 & -1\\-1 & -1 & 3\end{bmatrix}$ then find AB and hence find A^{-1}

Solution: $AB = \begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{bmatrix}\begin{bmatrix}3 & -1 & -1\\-1 & 3 & -1\\-1 & -1 & 3\end{bmatrix} = \begin{bmatrix}4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 4\end{bmatrix}$

Thus $AB = 4I$

Pre-Multiply by $A^{-1}AB = 4A^{-1}I \Rightarrow IB = 4A^{-1} \Rightarrow B = 4A^{-1} \Rightarrow A^{-1} = \frac{1}{4}B$

Thus $A^{-1} = \begin{bmatrix}\frac{3}{4} & -\frac{1}{4} & -\frac{1}{4}\\-\frac{1}{4} & \frac{3}{4} & -\frac{1}{4}\\-\frac{1}{4} & -\frac{1}{4} & \frac{3}{4}\end{bmatrix}$

Example:6 Find the inverse of the matrix $\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\\sin\alpha & \cos\alpha & 0\\0 & 0 & 1\end{bmatrix}$

Solution: Let $A = \begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\\sin\alpha & \cos\alpha & 0\\0 & 0 & 1\end{bmatrix}$

$|A| = \cos^2\alpha + \sin^2\alpha = 1$ (By expanding along third row)

$A_{11} = \cos\alpha, A_{12} = -\sin\alpha, A_{13} = 0$; $A_{21} = \sin\alpha, A_{22} = \cos\alpha, A_{23} = 0$; $A_{31} = 0, A_{32} = 0, A_{33} = 1$

$$\text{adj}(A)=\begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

Thus $A^{-1}=\frac{1}{|A|}\text{adj}(A)$

$$\Rightarrow A^{-1}=\begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

Example: -7: If A= $\begin{bmatrix}2 & 4\\ 3 & -2\end{bmatrix}$ then show that $A^{-1}=\frac{1}{16}A$

Solution: We have $|A|=-4-12=-16\neq 0$ Thus A^{-1} exists.

$$A^{-1}=\frac{1}{|A|}\text{adj}(A)=-\frac{1}{16}\begin{bmatrix}-2 & -4\\ -3 & 2\end{bmatrix}=\frac{1}{16}\begin{bmatrix}2 & 4\\ 3 & -2\end{bmatrix}=\frac{1}{16}A$$

Example -8: If $A=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$ then show that $A^T A^{-1}=\begin{bmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{bmatrix}$

Solution: $|A|=1+\tan^2 x\neq 0$, thus A^{-1} exists and is given by

$$A^{-1}=\frac{1}{|A|}\text{adj}(A)=\frac{1}{1+\tan^2 x}\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}$$

$$A^T=\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}$$

Thus $A^T A=\frac{1}{1+\tan^2 x}\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}=\frac{1}{1+\tan^2 x}\begin{vmatrix}1-\tan^2 x & -2\tan x\\ 2\tan x & 1-\tan^2 x\end{vmatrix}$

$$\Rightarrow A^T A^{-1}=\begin{bmatrix}\frac{1-\tan^2 x}{1+\tan^2 x} & -\frac{2\tan x}{1+\tan^2 x}\\ \frac{2\tan x}{1+\tan^2 x} & \frac{1-\tan^2 x}{1+\tan^2 x}\end{bmatrix}=\begin{bmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{bmatrix}$$

Example-9: For the matrix $\text{A}=\begin{bmatrix}2 & 3\\ 4 & -1\end{bmatrix}$, find the values of a ad b such that $A^2+aA+bI=0$.

Hence find A^{-1}

Solution: We have $A=\begin{bmatrix}2 & 3\\ 4 & -1\end{bmatrix}$

Therefore $A^2=\begin{bmatrix}2 & 3\\ 4 & -1\end{bmatrix}\begin{bmatrix}2 & 3\\ 4 & -1\end{bmatrix}=\begin{bmatrix}16 & 3\\ 4 & 13\end{bmatrix}$

As $A^2+aA+bI=0$

Therefore $\begin{bmatrix}16 & 3\\ 4 & 13\end{bmatrix}+\begin{bmatrix}2a & 3a\\ 4a & -a\end{bmatrix}+\begin{bmatrix}b & 0\\ 0 & b\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$$\Rightarrow\begin{bmatrix}16+2a+b & 3+3a\\ 4+4a & 13-a+b\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow 16+2a+b=0, 3+3a=0, 4+4a=0, 13-a+b=0$$

$$\Rightarrow a=-1 \text{ and } b=-14$$

Thus $A^2+aA+bI=0\Rightarrow A^2-A-14\text{I}=0$

Pre-Multiply by A^{-1} we get

$$A^{-1}A^2 - A^{-1}A - 14A^{-1}I = 0$$

$$\Rightarrow A - I - 14A^{-1} = 0 \quad \Rightarrow 14A^{-1} = A - I \quad \Rightarrow A^{-1} = \frac{1}{14}\left\{\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{14} & \frac{3}{14} \\ \frac{4}{14} & -\frac{2}{14} \end{bmatrix} = \begin{bmatrix} \frac{1}{14} & \frac{3}{14} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix}$$

Example-10: Find a 2×2 matrix B such that $B\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

Then, $|A| = 4 + 2 = 6 \neq 0$. Thus A^{-1} exits and is given by

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

Now $B\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \quad \Rightarrow BA = C$

Post-Multiply both side by A^{-1} we get $BAA^{-1} = CA^{-1}$ or $BI = CA^{-1}$ or

$$B = CA^{-1} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 24 & 12 \\ -6 & 6 \end{bmatrix}$$

Thus $B = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

Example:11 Find a matrix A such that $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} A \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Solution: Let $B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Thus we have $BAC = D$

Pre-Multiply by B^{-1} and post multiply by C^{-1} we get

$$B^{-1}BACC^{-1} = B^{-1}DC^{-1}$$

$$\Rightarrow IAI = B^{-1}DC^{-1}$$

$$\Rightarrow A = B^{-1}DC^{-1}$$

$$|B| = 1, |C| = 7$$

Thus $B^{-1} = \frac{1}{|B|}\text{adj}(B) = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

and $C^{-1} = \frac{1}{|C|}\text{adj}(C) = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$

Thus $A = \frac{1}{7}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$

Thus $A = \frac{1}{7}\begin{bmatrix} 1 & -7 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$

Thus $A=\frac{1}{7}\begin{bmatrix} 22 & -5 \\ -15 & 5 \end{bmatrix}=\begin{bmatrix} \frac{22}{7} & -\frac{5}{7} \\ -\frac{15}{7} & \frac{5}{7} \end{bmatrix}$

Exercise 4.6

1. Show that $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ is non singular.
2. Show that the matrix $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \\ 5 & 2 & 5 \end{bmatrix}$ is singular
3. Show that the matrix $\begin{bmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{bmatrix}$ is singular.
4. Find the value of λ such that the matrix $\begin{bmatrix} 2 & \lambda & 3 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ is singular.
5. Find the inverse of the matrix $\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$
6. Find the inverse of the matrix $\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$
7. Find the inverse of the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
8. Find the inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
9. Find the inverse of the matrix $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 5 \\ 2 & 0 & -1 \end{bmatrix}$
10. Find the inverse of the matrix $\begin{bmatrix} 3 & -1 & -2 \\ 2 & 1 & 3 \\ 4 & 1 & 2 \end{bmatrix}$
11. If $A=\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$ then verify that $(AB)^{-1}=B^{-1}A^{-1}$
12. If $A=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & -1 & 2 \\ 2 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$ then verify that $(AB)^{-1}=B^{-1}A^{-1}$
13. If $A=\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ then prove that $A^2-A-14I=0$ and hence find A^{-1}

14. If $A=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$ then find a and b such that $A^2+aA+bI=0$ and hence find A^{-1}

15. Find the inverse of the matrix $\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{bmatrix}$

16. For the matrix $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$, show that $A^3-6A^2+9A-4I=0$. Hence find A^{-1}

17. For the matrix $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$ show that $A^3-6A^2+5A+11I=0$

18. Show that the matrix $A=\begin{bmatrix}2 & 3\\ -1 & 4\end{bmatrix}$ is zero of the polynomial $f(x)=x^2-6x+11$ and hence find A^{-1}

19. Find the matrix X for which $\begin{bmatrix}3 & 2\\ 7 & 5\end{bmatrix}X=\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}$

20. Find the matrix X for which $X\begin{bmatrix}3 & -1\\ 2 & 4\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

21. Find the matrix A for which $\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}A\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}1 & 3\\ -1 & 0\end{bmatrix}$

22. Find the inverse of the matrix $A=\begin{bmatrix}a & b\\ c & \dfrac{1+bc}{a}\end{bmatrix}$ and show that $aA^{-1}=(A^2+bc+1)I-aA$

23. Given $A=\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$, compute A^{-1} and show that $2A^{-1}=9I-A$

24. If $A=\begin{bmatrix}4 & 5\\ 2 & 1\end{bmatrix}$, then show that $A-3I=2(I+3A^{-1})$

25. Given $A=\begin{bmatrix}5 & 0 & 4\\ 2 & 3 & 2\\ 1 & 2 & 1\end{bmatrix}, B^{-1}=\begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$. Compute $(AB)^{-1}$

26. Show that $\begin{bmatrix}1 & -\tan\dfrac{\theta}{2}\\ \tan\dfrac{\theta}{2} & 1\end{bmatrix}\begin{bmatrix}1 & \tan\dfrac{\theta}{2}\\ -\tan\dfrac{\theta}{2} & 1\end{bmatrix}^{-1}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

27. If $A=\dfrac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$, prove that $A^{-1}=A^T$

28. If $A=\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$, prove that $A^{-1}=A^3$

29. If $A=\begin{bmatrix}-1 & 2 & 0\\ -1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}$, show that $A^2=A^{-1}$

System of Linear Equations

Consider a system of linear equations $\begin{cases} a_1 x+b_1 y=c_1 \\ a_2 x+b_2 y=c_2 \end{cases}$.

Using matrix notations this can be written as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ or $AX=B$ where $A=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$,

$X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

If $|A|\neq 0$ then A^{-1} exists

Pre-Multiplying $AX=B$ by A^{-1} we get $A^{-1}AX=A^{-1}B \Rightarrow IX=A^{-1}B$

$\Rightarrow X=A^{-1}B$

Consider the system of equations $\begin{cases} a_1 x+b_1 y+c_1 z=d_1 \\ a_2 x+b_2 y+c_2 z=d_2 \\ a_3 x+b_3 y+c_3 z=d_3 \end{cases}$

This can be written as $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

or $AX=B$ where $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

If $|A|\neq 0$ then A^{-1} exists.

Pre-Multiplying $AX=B$ by A^{-1} we get $A^{-1}AX=A^{-1}B \Rightarrow IX=A^{-1}B \Rightarrow X=A^{-1}B$

Note: The system of equations $AX=B$ will have

(i) unique solution if $|A|\neq 0$ and the solution is given by $X=A^{-1}B$

(ii) infinitely many solutions if $\operatorname{adj}(A)B=0$

(iii) no solution if $\operatorname{adj}(A)B\neq 0$

Example-1: Solve the system of linear equations $\begin{cases} 2x+4y=10 \\ 3x-y=8 \end{cases}$

Solution: The given system of equations can be written as $\begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 10 \\ 8 \end{bmatrix}$

or $AX=B$ where $A=\begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, B=\begin{bmatrix} 10 \\ 8 \end{bmatrix}$

Now $|A|=\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix}=-2-12=-14\neq 0$, thus A^{-1} exists.

$A_{11}=-1, A_{12}=-3, A_{21}=-4, A_{22}=2$

Thus $\operatorname{adj}(A)=\begin{bmatrix} -1 & -4 \\ -3 & 2 \end{bmatrix}$

$\therefore \quad A^{-1}=\frac{1}{|A|}\operatorname{adj}(A)=-\frac{1}{14}\begin{bmatrix} -1 & -4 \\ -3 & 2 \end{bmatrix}$

Now $X=A^{-1}B \quad \Rightarrow X=-\frac{1}{14}\begin{bmatrix}-1 & -4\\ -3 & 2\end{bmatrix}\begin{bmatrix}10\\ 8\end{bmatrix}$

Thus $X=-\frac{1}{14}\begin{bmatrix}-42\\ -14\end{bmatrix}=\begin{bmatrix}3\\ 1\end{bmatrix} \quad \Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}3\\ 1\end{bmatrix}$

Hence solution of the system of equations is $x=3, y=1$

Example:2 Solve the system of equations $\begin{cases}3x-2y+z=3\\ x+2y-z=5\\ x-y+z=1\end{cases}$

Solution: The above system of equations can be written as $AX=B$ where

$$A=\begin{bmatrix}3 & -2 & 1\\ 1 & 2 & -1\\ 1 & -1 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, B=\begin{bmatrix}3\\ 5\\ 1\end{bmatrix}$$

$|A|=\begin{vmatrix}3 & -2 & 1\\ 1 & 2 & -1\\ 1 & -1 & 1\end{vmatrix}=3(2-1)-(-2)(1+1)+1(-1-2)=4\neq 0$. Thus A^{-1} exists.

Now

$A_{11}=1, A_{12}=-2, A_{13}=-3, A_{21}=1, A_{22}=2, A_{23}=1, A_{31}=0, A_{32}=4, A_{33}=8$

Thus $\operatorname{adj}(A)=\begin{bmatrix}1 & 1 & 0\\ -2 & 2 & 4\\ -3 & 1 & 8\end{bmatrix}$

Using $A^{-1}=\frac{1}{|A|}\operatorname{adj}(A)$ we get

$$A^{-1}=\frac{1}{4}\begin{bmatrix}1 & 1 & 0\\ -2 & 2 & 4\\ -3 & 1 & 8\end{bmatrix}$$

Using $X=A^{-1}B$ we get

$$X=\frac{1}{4}\begin{bmatrix}1 & 1 & 0\\ -2 & 2 & 4\\ -3 & 1 & 8\end{bmatrix}\begin{bmatrix}3\\ 5\\ 1\end{bmatrix}=\frac{1}{4}\begin{bmatrix}8\\ 8\\ 4\end{bmatrix}=\begin{bmatrix}2\\ 2\\ 1\end{bmatrix} \quad \Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2\\ 2\\ 1\end{bmatrix}$$

Thus solution of the given system of equations is $x=2, y=2, z=1$

Example-3: If $A=\begin{bmatrix}1 & 2 & 1\\ 1 & 0 & 3\\ 2 & -3 & 0\end{bmatrix}$ then find A^{-1} and hence solve the system of equations

$x+2y+z=7, x+3z=11, 2\text{x}-3\text{y}=1$

Solution: $|A|=\begin{vmatrix}1 & 2 & 1\\ 1 & 0 & 3\\ 2 & -3 & 0\end{vmatrix}=1(0+9)-2(0-6)+1(-3-0)=9+12-3=18\neq 0$

Thus A^{-1} exists.

$A_{11}=9, A_{12}=6, A_{13}=-3\,; A_{21}=-3, A_{22}=-2, A_{23}=7, A_{31}=6, A_{32}=-2, A_{33}=-2$

Thus $\operatorname{adj}(A)=\begin{bmatrix}9 & -3 & 6\\ 6 & -2 & -2\\ -3 & 7 & -2\end{bmatrix}$

$$\Rightarrow A^{-1}=\frac{1}{|A|}\text{adj}(A)=\frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

Now given system of equations can be expressed as $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$ or

$AX=B$ where $A=\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$.

$$\therefore X=A^{-1}B=\frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 2 \end{bmatrix}=\frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Hence solution of the system of equations is $x=2, y=1, z=3$

Example-5: If $A=\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, find A^{-1} and hence solve the system of equations $x+2y+z=4, -x+y+z=0, x-3\text{y}+z=2$

Solution: $A=\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

$$\therefore |A|=\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}=1(1+3)+1(2+3)+1(2-1)=10\neq 0$$

So, A is invertible.

$A_{11}=5, A_{12}=-5, A_{13}=1, A_{21}=2, A_{22}=0, A_{23}=-2, A_{31}=2, A_{32}=5, A_{33}=3$

$$\therefore \text{adj}(A)=\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1}=\frac{1}{|A|}\text{adj}(A)=\frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Now the given system of equations is expressible as $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

or, $A^T X=B$ where $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, B=\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Now, $|A^T|=|A|=10\neq 0$. So, the system of equations is consistent with a unique solution.

$X=(A^T)^{-1}B=(A^{-1})^T B$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}4&2&2\\-5&0&5\\1&-2&3\end{bmatrix}^T\begin{bmatrix}4\\0\\2\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}4&-5&1\\2&0&-2\\2&5&3\end{bmatrix}\begin{bmatrix}4\\0\\2\end{bmatrix}=\frac{1}{10}\begin{bmatrix}18\\4\\14\end{bmatrix}=\begin{bmatrix}\frac{9}{5}\\\frac{2}{5}\\\frac{7}{5}\end{bmatrix}$$

Hence $x=\frac{9}{5}, y=\frac{2}{5}, z=\frac{7}{5}$ is the solution of the given system of linear equations.

Example-6: Find the product of the matrices $\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$ and $\begin{bmatrix}3&-1&-1\\-1&3&-1\\-1&-1&3\end{bmatrix}$ and hence solve the system of equations $2x+y+z=4, x+2y+z=4, x+y+2z=4$

Solution: Let $A=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$ and $B=\begin{bmatrix}3&-1&-1\\-1&-3&-1\\-1&-1&3\end{bmatrix}$

$$AB=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\begin{bmatrix}3&-1&-1\\-1&-3&-1\\-1&-1&3\end{bmatrix}=\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}=4I$$

Pre-Multiplying by A^{-1} we get $A^{-1}AB=4A^{-1}I\Rightarrow IB=4A^{-1}\Rightarrow 4\mathrm{A}^{-1}=B\Rightarrow A^{-1}=\frac{1}{4}B$

Thus $A^{-1}=\frac{1}{4}\begin{bmatrix}3&-1&-1\\-1&3&-1\\-1&-1&3\end{bmatrix}$

Given system of equations can be written as $\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\4\\4\end{bmatrix}$ or $AX=C$ where

$$X=\begin{bmatrix}x\\y\\z\end{bmatrix}, C=\begin{bmatrix}4\\4\\4\end{bmatrix}$$

$$\therefore X=A^{-1}C=\frac{1}{4}\begin{bmatrix}3&-1&-1\\-1&3&-1\\-1&-1&3\end{bmatrix}\begin{bmatrix}4\\4\\4\\4\end{bmatrix}=\frac{1}{4}\begin{bmatrix}4\\4\\4\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Hence the solution of the given system of equations is $x=1, y=1, z=1$

Example-7: Solve the system of equations $2x-y+z=4, x+2y-z=3, 3x+y=7$ if consistent.

Solution: The given system of equations can be expressed as $\begin{bmatrix}2&-1&1\\1&2&-1\\3&1&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\3\\7\end{bmatrix}$

or $AX=B$ where $A=\begin{bmatrix}2 & -1 & 1\\1 & 2 & -1\\3 & 1 & 0\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}4\\3\\7\end{bmatrix}$

$|A|=2(0+1)+1(0+3)+1(1-6)=2+3-5=0 \quad \Rightarrow A^{-1}$ does not exist.

Thus system of equations does not have unique solution.

$A_{11}=1, A_{12}=-3, A_{13}=-5, A_{21}=1, A_{22}=-3, A_{23}=-5, A_{31}=-1, A_{32}=3, A_{33}=5$

Thus $\operatorname{adj}(A)=\begin{bmatrix}1 & 1 & -1\\-3 & -3 & 3\\-5 & -5 & 5\end{bmatrix}$

$\Rightarrow \operatorname{adj}(A)B=\begin{bmatrix}1 & 1 & -1\\-3 & -3 & 3\\-5 & -5 & 5\end{bmatrix}\begin{bmatrix}4\\3\\7\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

Thus system of equations has infinitely many solutions.

To find the solutions let us put $z=k$ in the first two equations. Thus we get

$2x-y+k=4, x+2y-k=3$

or $2x-y=4-k, x+2\text{y}=3+k$ This can be written as

$\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4-k\\3+k\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}^{-1}\begin{bmatrix}4-k\\3+k\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{5}\begin{bmatrix}2 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}4-k\\3+k\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{5}\begin{bmatrix}11-k\\2+3\text{k}\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{11-k}{5}\\\frac{2+3k}{5}\end{bmatrix}$

Thus solution of the given system of equations is $x=\frac{11-k}{5}, y=\frac{2+3k}{5,} z=k$ where k is any real number.

Example-8: Show that the system of equations $2x+y+z=4, x+y+z=1, 3x+2y+2z=4$ is inconsistent.

Solution: The given system of equations can be written as $\begin{bmatrix}2 & 1 & 1\\1 & 1 & 1\\3 & 2 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\1\\4\end{bmatrix}$

or $AX=B$ where $A=\begin{bmatrix}2 & 1 & 1\\1 & 1 & 1\\3 & 2 & 2\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}4\\1\\4\end{bmatrix}$

$|A|=2(2-2)-1(2-3)+1(2-3)=0$. Thus A is singular matrix. Thus A^{-1} does not exist.

Now $A_{11}=0, A_{12}=1, A_{13}=-1, A_{21}=0, A_{22}=1, A_{23}=-1, A_{31}=0, A_{32}=-1, A_{33}=1$

Thus $\operatorname{adj}(A)=\begin{bmatrix}0 & 0 & 0\\1 & 1 & -1\\-1 & -1 & 1\end{bmatrix}$

$$\Rightarrow \operatorname{adj}(A)B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

As $|A|=0$ and $\operatorname{adj}(A)B \neq 0$, therefore the system of equations do not have any solution.
Thus the given system of equations is inconsistent.

Homogeneous system of equations: A system of equations is said to be homogeneous if it can be written as $AX=B$

Trivial Solution: A solution in which each variable is zero is known as trivial solution. Every system homogeneous linear equations always has trivial solution. A homogeneous system of linear equations will have trivial solution only if $|\mathrm{A}|=0$

Non-trivial Solution: A homogeneous system of linear equations $\mathrm{AX}=\mathrm{B}$ will have non-trivial solution if $|\mathrm{A}|\neq 0$

Example-9: Solve the following homogeneous system of linear equations
$2\mathrm{x}-y+z=0, x+y-z=0, x-y=0$

Solution: The given system of equations can be written as $AX=0$ where

$$A=\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Now $|A|=2(0-1)+1(0+1)+1(-1-1)=-2+1-2=-3\neq 0$
Thus the given system of equations has trivial solution.
Thus solution is $x=0, y=0, x=0$

Exercise 4.7

Solve the following system of linear equations by matrix method

1. $5x+7y+2$
   $4x+6y+3=0$

2. $3x+4y-5=0$
   $x-y+3=0$

3. $x+y+z=3$
   $2x-y+z=-1$
   $2x+y-3\mathrm{z}=-9$

4. $6x-12y+25z=4$
   $4x+15y-20z=3$
   $2x+18y+15z=10$

5. $3x+4y+7z=16$
   $2x-y+3z=19$
   $x+2y-3\mathrm{z}=25$

$$\frac{2}{x}-\frac{3}{y}+\frac{3}{z}=10$$
6. $$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=10$$
$$\frac{3}{x}-\frac{1}{y}+\frac{2}{z}=13$$

7. $$\begin{aligned} 3x+4y+2z&=8 \\ 2y-3z&=3 \\ x-2y+6z&=-2 \end{aligned}$$

8. $$\begin{aligned} 2x+6y&=2 \\ 3x-z&=-8 \\ 2x-y+z&=-3 \end{aligned}$$

9. $$\begin{aligned} 8x+4y+3z&=18 \\ 2x+y+z&=5 \\ x+2y+z&=5 \end{aligned}$$

10. $$\begin{aligned} 2x+y+z&=4 \\ x+2y+z&=4 \\ x+y+2z&=4 \end{aligned}$$

11. If $A=\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ are two square matrices, find AB and hence solve the system of equations $x-y=3, 2x+3y+4z=17, y+2z=7$

12. If $A=\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$ find A^{-1} and hence solve the system of equations $2x-3y+5z=11, 3x+2y-4z=-5, x+y+2z=-3$

13. Find A^{-1} if $A=\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$. Hence solve the system of equations $x+2y+5z=10, x-y-z=-2, 2x+3y-z=-11$

14. Find A^{-1} if $A=\begin{bmatrix} 4 & 5 & 2 \\ -5 & -4 & 2 \\ -2 & 2 & 8 \end{bmatrix}$ and hence solve the system of equations $4x-5y-2z=2, 5x-4y+2z=-2, 2x+2y+8z=-1$

15. Solve the following system of equations if consistent $2x+y+z=4, x+y+z=1, 3x+2y+2z=5$

16. Show that the system of equations $3x+y+z=1, x+y-z=2, 4x+2y=2$

17. Solve the following homogeneous\system of linear equations

(i) $$\begin{aligned} 2x-y+z&=0 \\ 3x+2y-z&=0 \\ x+4y+3z&=0 \end{aligned}$$ (ii) $$\begin{aligned} 2x-y-z&=0 \\ x+y+z&=0 \\ 3x+2y-z&=0 \end{aligned}$$ (iii) $$\begin{aligned} x+y+z&=0 \\ x-y-5z&=0 \\ x+2y+4z&=0 \end{aligned}$$

(iv) $\begin{aligned} x+y-z&=0 \\ x-2y+z&=0 \\ 3x+6y-5z&=0 \end{aligned}$ (v) $\begin{aligned} 2x+y+z&=0 \\ xx+2y+z&=0 \\ x+y+2z&=0 \end{aligned}$ (vi) $\begin{aligned} 2x-y+2z&=0 \\ 5x+3y-z&=0 \\ x+5y-5z&=0 \end{aligned}$