INT 1: If $\frac{df(x)}{dx}=F(x)$ then we define $\int F(x)dx=f(x)+c$. For example

1. $\frac{d}{dx}(\sin x)=\cos x \Rightarrow \int \cos x dx=\sin x+c$
2. $\frac{d}{dx}(\tan x)=\sec^2 x \Rightarrow \int \sec^2 x dx=\tan x+c$

Note: $\int F(x)dx$ is read as integration of $F(x)$ with respect to x

Here c is constant of integration.

INT 2: Memory Tips:

1. $\int x^n dx=\frac{x^{n+1}}{n+1}+c,\ (n+1\neq 0)\quad \therefore \int dx=x+c$

2. $\int (ax+b)^n dx=\frac{(ax+b)^{n+1}}{(n+1)a}+c\,(n+1\neq 0)$

3. $\int \frac{dx}{x}=\log|x|+c$

4. $\int \frac{dx}{ax+b}=\frac{\log|ax+b|}{a}+c$

5. $\int a^x dx=\frac{a^x}{\log a}+c$ and $\int e^x dx=e^x+c$

6. $\int \sin x dx=-\cos x+c$

7. $\int \cos x dx=\sin x+c$

8. $\int \tan x dx=-\log|\cos x|+c=\log|\sec x|+c$

9. $\int \cot x dx=\log|\sin x|+c$

10. $\int \sec x\tan x dx=\sec x+c$

11. $\int \cos ecx\cot x dx=-\cos ecx+c$

12. $\int \sec^2 x dx=\tan x+c$

13. $\int \cos ec^2 x dx=-\cot x+c$

14. $\int \sec x dx=\log|\sec x+\tan x|+c=\log\left|\tan(\frac{\pi}{4}+\frac{x}{2})\right|+c$

15. $\int \cos ecx dx=\log|\cos ecx-\cot x|+c=\log\left|\tan\frac{x}{2}\right|+c$

16. $\int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c=-\cos^{-1}x+c$

17. $\int \frac{dx}{1+x^2}=\tan^{-1}x+c=-\cot^{-1}x+c$

18. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c = -\operatorname{cosec}^{-1} x + c$

19. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

20. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a-x}{a+x}\right| + c$

21. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

22. $\int \frac{dx}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + c$

23. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c$

24. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + c$

25. $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

26. $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

27. $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

28. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

29. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\left[a\cos bx + b\sin bx\right] + c$

30. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\left[a\sin bx - b\cos bx\right] + c$

Fundamental Results of Integration:

1. $\int kf(x)\,dx = k\int f(x)dx$
2. $\int\{f(x)+g(x)\}\,dx = \int f(x)\,dx + \int g(x)\,dx$
3. $\int\{f(x)-g(x)\}\,dx = \int f(x)\,dx - \int g(x)\,dx$
4. $\int f(x)dx = F(x)+c \Rightarrow \int f(ax+b)\,dx = \frac{F(ax+b)}{a} + C$

Method of Substitution:

To evaluate integral of the types $\int (f(x))^n f'(x)\,dx$ put

$f(x)=t$ so that $f'(x)dx = dt$ and $\int (f(x))^n f'(x)dx = \int t^n dt = \frac{t^{n+1}}{n+1} + c$.

Hence $\int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + C$ provided $n \neq -1$

If integral is of the type

$\int \frac{f(x)}{f'(x)}dx$ the put $f(x) = t$ so that $f'(x)dx = dt$ and $\int \frac{f(x)}{f'(x)}dx = \int \frac{dt}{t} = \log|t| + c$.

Hence $\int \frac{f(x)}{f'(x)}dx = \log|f(x)| + C$

Examples:

1. Evaluate $\int \sin^3 x \cos x dx$

   Solution: Put $\sin x = t \Rightarrow \cos x dx = dt$

   $$\Rightarrow \int \sin^3 x \cos x dx = \int t^3 dt = \frac{t^{3+1}}{3+1} + C = \frac{t^4}{4} + C = \frac{\sin^4 x}{4} + C$$

2. $\int \frac{(x+1)e^x}{\cos^2(xe^x)}dx$

   Solution: Put $xe^x = t \Rightarrow x\frac{d}{dx}e^x + e^x\frac{d}{dx}x = \frac{dt}{dx}$

   $$\Rightarrow xe^x + e^x = \frac{dt}{dx} \Rightarrow (x+1)e^x dx = dt$$

   $$\Rightarrow \int \frac{(x+1)e^x}{\cos^2(xe^x)}dx = \int \frac{1}{\cos^2 t}dt = \int \sec^2 t dt = \tan t + C = \tan(xe^x) + C$$

3. $\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x}dx$

   Solution: Put $a^2 \sin^2 x + b^2 \cos^2 x = t$. Differentiate both sides with respect to x we get $(a^2 - b^2)\sin 2x = \frac{dt}{dx} \Rightarrow \sin 2x dx = \frac{1}{(a^2 - b^2)}dt$

   $$\therefore \int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x}dx = \frac{1}{(a^2 - b^2)}\int \frac{1}{t}dt$$

   $$= \frac{1}{(a^2 - b^2)}\log|t| + C = \frac{1}{(a^2 - b^2)}\log\left|a^2 \sin^2 x + b^2 \cos^2 x\right| + C$$

4. $\int \tan x dx$

   Solution: $\int \tan x dx = \int \frac{\sin x}{\cos x}dx$

   Put $\cos x = t \Rightarrow -\sin x = \frac{dt}{dx} \Rightarrow \sin x dx = -dt$

   $$\therefore \int \tan x dx = -\int \frac{1}{t}dt = -\log|t| + C = -\log|\cos x| + C = \log|\sec x| + C$$

Integral of the Type: $\int (ax+b)\sqrt{px+q}\,dx$ i.e. $\int \text{Linear}\sqrt{\text{Linear}}\,dx$

Method: Put $\sqrt{\text{Linear}} = t$ i.e. $px+q=t^2 \Rightarrow pdx = 2tdt \Rightarrow dx = \frac{2t}{p}dt$

$$\therefore \int (ax+b)\sqrt{px+q}dx = \int \left\{ a\left(\frac{t^2-q}{p}\right)+b \right\} \frac{2t^2}{p}dt$$

$$= \frac{2}{p^2}\int \left(at^4 - aqt^2 + bpt^2\right)dt = \frac{a}{p^2}\left(\frac{2}{5}t^5 - \frac{2}{3}qt^3\right) + \frac{2bt^3}{3p}$$

$$= \frac{2a}{5p^2}t^5 - 2\left[\frac{aq-bp}{3p^2}\right]t^3 + c$$

Hence $\boxed{\int (ax+b)\sqrt{px+q}dx = \frac{2a}{5p^2}(px+q)^{\frac{5}{2}} - 2\left[\frac{aq-bp}{3p^2}\right](px+q)^{\frac{3}{2}} + C}$

Example: Evaluate $\int (3x+4)\sqrt{2x+5}dx$

Solution: Put $2x+5=t^2 \Rightarrow 2dx = 2tdt \Rightarrow dx = tdt$

$$\therefore \int (3x+4)\sqrt{2x+5}dx = \int \left\{ 3\left(\frac{t^2-5}{2}\right)+4 \right\} t^2dt = \int \left(\frac{3t^4}{2} - \frac{7}{2}t^2\right)dt$$

$$= \frac{3t^5}{10} - \frac{7t^3}{6} + c = \frac{3}{10}(2x+5)^{\frac{5}{2}} - \frac{7}{6}(2x+5)^{\frac{3}{2}} + c$$

Integral of the type: $\int \frac{ax+b}{\sqrt{px+q}}dx$

Method: Let $ax+b = \alpha(px+q)+\beta \Rightarrow \alpha = \frac{a}{p}$ & $\beta = \frac{bp-aq}{p}$

$$\therefore \int \frac{ax+b}{\sqrt{px+q}}dx = \int \frac{\alpha(px+q)+\beta}{\sqrt{px+q}}dx = \alpha\int \sqrt{px+q}dx + b\int \frac{1}{\sqrt{px+q}}dx$$

$$= \frac{2a}{3p^2}(px+q)^{\frac{3}{2}} + \frac{2(bp-aq)}{p^2}(px+q)^{\frac{1}{2}} + C$$

Example: Evaluate $\int \frac{x+2}{\sqrt{2x+3}}dx$

Solution: Put $2x+3=t^2 \Rightarrow 2dx = 2tdt \Rightarrow dx = tdt$

$$\Rightarrow \int \frac{x+2}{\sqrt{2x+3}}dx = \int \frac{\frac{t^2-3}{2}+2}{t}tdt = \int \frac{t^2+1}{2}dt = \frac{t^3}{6} + \frac{1}{2}t + c = \frac{(2x+3)^{\frac{3}{2}}}{6} + \frac{\sqrt{2x+3}}{2} + c$$

Integral of the type: $\int \frac{e^{kx}+e^{-kx}}{e^{kx}-e^{-kx}}dx$

Method: Divide numerator and denominator of the integrand by $e^{\frac{kx}{2}}$

$$\Rightarrow \int \frac{e^{kx}+1}{e^{kx}-1}dx = \int \frac{e^{kx/2}+e^{-kx/2}}{e^{kx/2}-e^{-kx/2}}dx$$

Put $e^{kx/2}-e^{-kx/2}=t \Rightarrow \frac{k}{2}\left(e^{kx/2}+e^{-kx/2}\right)dx = dt \Rightarrow \left(e^{kx/2}+e^{-kx/2}\right)dx = \frac{2}{k}dt$

$$\therefore \int \frac{e^{kx}+1}{e^{kx}-1}dx = \frac{2}{k}\int \frac{1}{t}dt = \frac{2}{k}\log|t|+c = \frac{2}{k}\log\left|e^{kx}-e^{-kx}\right|+c$$

Example: Evaluate $\int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx$

Solution: $\int \frac{e^{2x}+1}{e^{2x}-1}dx = \int \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}dx$

Put $\left(e^{x}-e^{-x}\right)=t \Rightarrow \left(e^{x}+e^{-x}\right)dx = dt$

$$\Rightarrow \int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx = \int \frac{1}{t}dt = \log|t|+c = \log\left|e^{x}-e^{-x}\right|+c$$

Integral of the type $\int \sqrt{ae^{x}+b}$

Method: Put $ae^{x}+b=t^2 \Rightarrow ae^{x}dx = 2tdt \Rightarrow dx = \frac{2t}{ae^{x}}dt \Rightarrow dx = \frac{2t}{t^2-b}dt$

$$\therefore \int \sqrt{ae^{x}+b} = \int t\times\frac{2t}{t^2-b}dt = 2\int \frac{t^2}{t^2-b}dt = 2\int \frac{t^2-b+b}{t^2-b}dt = 2\int dt + 2b\int \frac{1}{t^2-\left(\sqrt{b}\right)^2}dt$$

$$= 2t + 2b\times\frac{1}{2\sqrt{b}}\log\left|\frac{t-\sqrt{b}}{t+\sqrt{b}}\right|+c = 2\sqrt{ae^{x}+b}+\sqrt{b}\log\left|\frac{\sqrt{ae^{x}+b}-\sqrt{b}}{\sqrt{ae^{x}+b}+\sqrt{b}}\right|+c$$

Example: Evaluate $\int \sqrt{2e^{x}+3}dx$

Solution: Put $2e^{x}+3=t^2 \Rightarrow 2e^{x}dx = 2tdt \Rightarrow \left(t^2-3\right)dx = 2tdt \Rightarrow dx = \frac{2t}{t^2-3}dt$

$$\Rightarrow \int \sqrt{2e^{x}+3}dx = \int t\times\frac{2t}{t^2-3}dt = 2\int \frac{t^2-3+3}{t^2-3}dt = \int 2dt + 6\int \frac{1}{t^2-\left(\sqrt{3}\right)^2}dt = 2t + \frac{6}{2\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+c$$

$$= 2\sqrt{2e^{x}+3}+\sqrt{3}\log\left|\frac{\sqrt{2e^{x}+3}-\sqrt{3}}{\sqrt{2e^{x}+3}+\sqrt{3}}\right|+c$$

Integral of the type: $\int \frac{1}{\sqrt{ae^{x}+b}}dx$

Method: Put $ae^{x}+b=t^2 \Rightarrow ae^{x}dx = 2tdt \Rightarrow dx = \frac{2t}{ae^{x}}dt \Rightarrow dx = \frac{2t}{t^2-b}dt$

$$\therefore \int \frac{1}{\sqrt{ae^x+b}}dx = \int \frac{1}{t}\times\frac{2t}{t^2-b}dt = 2\int\frac{1}{t^2-b}dt = \frac{2}{2\sqrt{b}}\log\left|\frac{t-\sqrt{b}}{t+\sqrt{b}}\right| + c = \frac{1}{\sqrt{b}}\log\left|\frac{\sqrt{ae^x+b}-\sqrt{b}}{\sqrt{ae^x+b}+\sqrt{b}}\right| + c$$

Example: Evaluate $\int \frac{1}{\sqrt{2e^x+3}}dx$

Solution: Put $2e^x+3=t^2 \Rightarrow 2e^x dx = 2tdt \Rightarrow \left(t^2-3\right)dx = 2tdt \Rightarrow dx = \frac{2t}{t^2-3}dt$

$$\Rightarrow \int \frac{1}{\sqrt{2e^x+3dx}} = \int \frac{1}{t}\times\frac{2t}{t^2-3}dt = 2\int\frac{1}{t^2-3}dt = \frac{2}{2\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + c = \frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{2e^x+3}-\sqrt{3}}{\sqrt{2e^x+3}+\sqrt{3}}\right| + c$$

Integration involving trigonometric functions:

Integral of type $\int \sin^m xdx$ **&** $\int \cos^m xdx$ if $m \le 4$

Use the following results

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\cos^3 x = \frac{3\cos x + \cos 3x}{4}$$

Example1. Evaluate $\int \sin^2 3xdx$

Solution $\int \sin^2 3x = \int \frac{1-\cos 6x}{2}dx = \frac{1}{2}x - \frac{\sin 6x}{12} + c$

Example2. Evaluate $\int \sin^3 2xdx$

Solution $\int \sin^3 2xdx = \int \frac{3\sin 2x - \sin 6x}{4}dx = -\frac{3}{8}\cos 2x + \frac{1}{24}\cos 6x + c$

Example3. Evaluate $\int \cos^2 xdx$

Solution $\int \cos^2 xdx = \int \frac{1+\cos 2x}{2}dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

Example4. Evaluate $\int \cos^3 xdx$

Solution $\int \cos^3 xdx = \int \frac{3\cos x + \cos 3x}{4}dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + c$

Example5. Evaluate $\int \cos^4 2xdx$

Solution

$$\int \cos^4 2x dx = \int \left(\frac{1+\cos 4x}{2}\right)^2 dx = \int \frac{1+2\cos 4x+\cos^2 4x}{4} dx = \frac{1}{4}\int\left(1+2\cos 4x+\frac{1+\cos 8x}{2}\right)dx$$

$$=\frac{1}{8}\int\left(\cos 8x+4\cos 4x+3\right)dx = \frac{1}{64}\sin 8x+\frac{1}{8}\sin 4x+\frac{3}{8}x+c$$

Example6. Evaluate $\int \cos^4 x dx$

Solution

$$\int \sin^4 x dx = \int\left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int\left(1-2\cos 2x+\cos^2 2x\right)dx = \frac{1}{4}\int\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$$

$$=\frac{1}{8}\int\left(\cos 4x-4\cos 2x+3\right)dx = \frac{\sin 4x}{32}-\frac{\sin 2x}{4}+\frac{3x}{8}+c$$

Integral of the type $\int \sin^n x \cos^n x dx, n \le 4$

Method Use the result $\sin x\cos x = \frac{1}{2}\sin 2x$

$\Rightarrow \int \sin^n x\cos^n x dx = \int\left(\sin x\cos x\right)^n dx = \int\left(\frac{\sin 2x}{2}\right)^n = \frac{1}{2^n}\int \sin^n 2x dx$**.** This integral can be evaluated by the method given in the previous article.

Integral of the type $\int \sin^n x dx$ or $\int \cos^n x dx$

Case1. When n is even

$$\cos^n x = \frac{1}{2^{n-1}}\left\{\cos nx + {}^nC_1\cos(n-2)x + {}^nC_2 Cos(n-4)x+\cdots+\frac{1}{2}.{}^nC_{\frac{n}{2}}\right\}$$

$$\sin^n x = \frac{(-1)^{\frac{n}{2}}}{2^{n-1}}\left(\cos nx - {}^nC_1\cos(n-2)x + {}^nC_2\cos(n-4)x+\cdots+(-1)^{\frac{n}{2}}\cdot\frac{1}{2}\cdot{}^nC_{\frac{n}{2}}\right)$$

Case2. When n is odd

$$\cos^n x = \frac{1}{2^{n-1}}\left\{\cos nx + {}^nC_1\cos(n-2)x + {}^nC_2 Cos(n-4)x+\cdots+{}^nC_{\frac{n-1}{2}}\cos x\right\}$$

$$\sin^n x = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}}\left(\sin nx - {}^nC_1\sin(n-2)x + {}^nC_2\sin(n-4)x+\cdots+(-1)^{\frac{n}{2}}\,{}^nC_{\frac{n-1}{2}}\sin x\right)$$

Example1. Evaluate $\int \sin^8 x dx$

Solution $\sin^8 x = \frac{(-1)^4}{128}\left(\cos 8x - {}^8C_1\cos 6x + {}^8C_2\cos 4x - {}^8C_3\cos 2x+\frac{1}{2}\times{}^8C_4\right)$

$$\Rightarrow \sin^8 x = \frac{1}{128}\left(\cos 8x - 8\cos 6x + 28\cos 4x - 56\cos 2x + 35\right)$$

$$\Rightarrow \int \sin^8 x dx = \frac{1}{128}\left(\frac{\sin 8x}{8} - \frac{4\sin 6x}{3} + 7\sin 4x - 28\sin 2x + 35x\right) + c$$

Example2. Evaluate $\int \sin^6 x dx$

Solution $\sin^6 x = \frac{-1}{32}(\cos 6x - 6\cos 4x + 15\cos 2x - 10)$

$$\Rightarrow \int \sin^6 x dx = \frac{-1}{32}\left(\frac{\sin 6x}{6} - \frac{3\sin 4x}{2} + \frac{15\sin 2x}{2} - 10x\right) + c$$

Example3. Evaluate $\int \cos^8 x dx$

Solution $\cos^8 x = \frac{1}{128}(\cos 8x + 8\cos 6x + 28\cos 4x + 56\cos 2x + 35)$

$$\Rightarrow \int \cos^8 x dx = \frac{1}{128}\left(\frac{\sin 8x}{8} + \frac{4\sin 6x}{3} + 7\sin 4x + 28\sin 2x + 35x\right) + c$$

Example4. Evaluate $\int \cos^6 x dx$

Solution $\cos^6 x = \frac{1}{32}(\cos 6x + 6\cos 4x + 15\cos 2x + 10)$

$$\int \cos^6 x dx = \frac{1}{32}\left(\frac{\sin 6x}{6} + \frac{3\cos 4x}{2} + \frac{15\sin 2x}{2} + 10x\right) + c$$

Example5. Evaluate $\int \sin^5 x dx$

Solution $\sin^5 x = \frac{1}{16}(\sin 5x - 5\sin 3x + 10\sin x)$

$$\Rightarrow \int \sin^5 x dx = \frac{1}{16}\left(-\frac{\cos 5x}{5} + \frac{5\cos 3x}{3} - 10\cos x\right) + c$$

Example6. Evaluate $\int \cos^7 x dx$

Solution $\cos^7 x = \frac{1}{64}(\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x)$

$$\Rightarrow \int \cos^7 x dx = \frac{1}{64}\left(\frac{\sin 7x}{7} + \frac{7\sin 5x}{5} + 7\sin 3x + 35\sin x\right) + c$$

Integral of the type $\int \sin^m x \cos^n x dx$ by using De Moivre's Theorem

Example1 Evaluate $\int \cos^7 x \sin^5 x dx$

Solution. Let $z = \cos x + i\sin x \Rightarrow z^{-1} = \cos x - i\sin x$

$\Rightarrow 2\cos x = z + z^{-1}$ & $2i\sin x = z - z^{-1}$

$\Rightarrow (2\cos x)^7 (2i\sin x)^5 = (z + z^{-1})^7 (z - z^{-1})^5 = (z^2 - z^{-5})^5 (z + z^{-1})^2$

$\Rightarrow (2\cos x)^7 (2i\sin x)^5 = (z^{10} - 5z^6 + 10z^2 - 10z^{-2} + 5z^{-6} - z^{-10})(z^2 + 2 + z^{-2})$

$= (z^{12} - z^{-12}) + 2(z^{10} - z^{-10}) - 4(z^8 - z^{-8}) - 10(z^6 - z^{-6}) + 5(z^4 - z^{-4}) + 20(z^2 - z^{-2})$

Since $z^p - z^{-p} = 2i\sin px$ we have

$\cos^7 x\sin^5 x = 2^{-11}\left(\sin 12x + 2\sin 10x - 4\sin 8x - 10\sin 6x + 5\sin 4x + 20\sin 2x\right)$

$$\int \cos^7 x\sin^5 x dx = 2^{-11}\left\{-\frac{\cos 12x}{12} - \frac{2\sin 10x}{10} + \frac{4\cos 8x}{8} + \frac{10\cos 6x}{6} - \frac{5\cos 4x}{4} - \frac{20\cos 2x}{2}\right\} + c$$

Integral of the type $\int \sin^m x\cos^n x dx$ when

- If $m = 2k+1$ then put $t = \cos x$
- If $n = 2k+1$ then $t = \sin x$
- If $m \,\&\, n = 2k+1$ then put $t = \cos x$ or $t = \sin x$
- If $m \,\&\, n \neq 2k+1$, then
  - If $m + n = -2k$, put $t = \tan x$
  - If $m + n = 2k$ then express the integrand as algebraic sum of $\sin \&\cos$ of multiple angles, using De-Moivre’s Theorem
  - If $m + n = -(2k+1)$, then multiply by suitable power of $\left(\cos^2 x + \sin^2 x\right)$

Example1. Evaluate $\int \sin^5 x\cos^4 x dx$

Solution $I = \int \sin^5 x\cos^4 x dx = \int \sin^4 x\cos^4 x\sin x dx = \int \left(1-\cos^2 x\right)^2 \cos^4 x\sin x dx$

Put $\cos x = t \Rightarrow \sin x dx = -dt$

$$\Rightarrow I = \int \left(1-t^2\right)^2 t^4\left(-dt\right) = -\int \left(t^4 - 2t^6 + t^8\right)dt = -\left(\frac{t^5}{5} - \frac{2t^7}{7} + \frac{t^9}{9}\right) + c$$

$$= -\frac{\sin^5 x}{5} + \frac{2\sin^7 x}{7} - \frac{\sin^9 x}{9} + c$$

Example2. Evaluate $\int \frac{\cos^3 x}{\sqrt{\sin^7 x}} dx$

Solution Let $I = \int \frac{\cos^3 x}{\sqrt{\sin^7 x}} dx$, here $m = -\frac{7}{2}, n = 3 \therefore$ Put $\sin x = t \Rightarrow \cos x dx = dt$

$$\therefore I = \int \frac{\left(1-\sin^2 x\right)\sin x}{\sin^{\frac{7}{2}} x} dx = \int \frac{\left(1-t^2\right)}{t^{7/2}} dt = \int \left(t^{-7/2} - t^{-3/2}\right)dt$$

$$= -\frac{2}{5}t^{-5/2} + 2t^{-1/2} + C = -\frac{2}{5}\cos ec^{5/2}x + 2\sqrt{\cos ecx} + C$$

Example3. Evaluate $\int \sec^{7/3} x\cos ec^{5/3} dx$

Solution Let $I = \int \cos^{-7/3} \sin^{-5/3} dx$ .Here $m = -\frac{5}{3}, n = -\frac{7}{2} \Rightarrow m + n = -4$

Therefore put $\tan x = t \Rightarrow \sec^2 x dx = dt \Rightarrow dx = \frac{1}{1+\tan^2 x} dx = \frac{1}{1+t^2} dt$

Let $I=\int\left(1+t^2\right)^{7/6}\frac{\left(1+t^2\right)^{5/6}}{t^{5/3}}\frac{dt}{\left(1+t^2\right)}=\int\frac{\left(1+t^2\right)}{t^{5/3}}dt=\int\left(t^{-5/3}+t^{1/3}\right)dt=-\frac{3}{2}t^{-2/3}+\frac{3}{4}t^{4/3}+c$

$=-\frac{3}{2}\cot^{2/3}x+\frac{3}{4}\tan^{4/3}x+c$

Integral of the form: $I=\int R(\sin x,\cos x)\,dx$ where R is a rational function of $\sin x$ & $\cos x$, are transformed into integrals of a rational function by the substitution $\tan\left(\frac{x}{2}\right)=t$

In some special cases the integral can be simplified by

- Substituting $\sin x=t$ if $I=\int R(\sin x)\cos x\,dx$
- Substituting $\cos x=t$ if $I=\int R(\cos x)\sin x\,dx$
- Substituting $\tan x=t$ if $I=\int R(\tan x)\,dx$
- Substituting $\tan x=t$, if $R(-\sin x,-\cos x)=R(\sin x,\cos x)$
- Substituting $\cos x=t$ if $R(-\sin x,\cos x)=-R(\sin x,\cos x)$
- Substituting $\sin x=t$ if $R(\sin x,-\cos x)=-R(\sin x,\cos x)$

Example1. $\int\frac{dx}{\sin x(2+\cos x-2\sin x)}$

Solution Let $I=\int\frac{dx}{\sin x(2+\cos x-2\sin x)}$

Put $\tan\left(\frac{x}{2}\right)=t\Rightarrow\frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx=dt\Rightarrow\frac{1}{2}\left(1+t^2\right)dx=dt\Rightarrow dx=\frac{2}{1+t^2}dt$

$$I=\int\frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2}\left(2+\frac{1-t^2}{1+t^2}-\frac{4t}{1+t^2}\right)}=\int\frac{\left(1+t^2\right)}{t\left(t^2-4t+3\right)}dt=\int\frac{\left(1+t^2\right)}{t(t-1)(t-3)}dt$$

$$=\frac{1}{3}\int\frac{1}{t}dt+\frac{5}{3}\int\frac{1}{t-3}dt-\int\frac{1}{t-1}dt=\frac{1}{3}\log\left|\tan\left(\frac{x}{2}\right)\right|+\frac{5}{3}\log\left|\tan\left(\frac{x}{2}\right)-3\right|-\log\left|\tan\frac{x}{2}-1\right|+c$$

Example2. $\int\frac{dx}{\sin x\left(2\cos^2 x-1\right)}$

Solution $I=\int\frac{dx}{\sin x\left(2\cos^2 x-1\right)}$ In this expression if we substitute $-\sin x$ for $\sin x$, then it will change its sign. So put $\cos x=t\Rightarrow\sin x\,dx=-dt$

$$\Rightarrow I=\int\frac{\sin x\,dx}{\sin^2 x\left(2\cos^2 x-1\right)}=\int\frac{\sin x\,dx}{\left(1-\cos^2 x\right)\left(2\cos^2 x-1\right)}=-\int\frac{dt}{\left(1-t^2\right)\left(2t^2-1\right)}$$

$$= 2\int \frac{dt}{1-2t^2} - \int \frac{dt}{1-t^2} = \frac{1}{\sqrt{2}}\log\left|\frac{1+t\sqrt{2}}{1-t\sqrt{2}}\right| - \frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + C = \frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x}\right| - \frac{1}{2}\log\left|\frac{1+\cos x}{1-\cos x}\right| + c$$

Example3 $\int \frac{1}{5+4\cos x}dx$

Solution

$$I = \int \frac{1}{5+4\cos x}dx = \int \frac{1}{5+4\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}dx = \int \frac{\tan^2(x/2)}{\tan^2(x/2)+9}dx = \int \frac{\sec^2(x/2)}{\tan^2(x/2)+9}$$

Put $\tan\left(\frac{x}{2}\right) = t \Rightarrow \sec^2\left(\frac{x}{2}\right)dx = dt$

$$\Rightarrow I = \int \frac{2dt}{t^2+9} = \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right) + c = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{x}{2}\right)\right) + c$$

Example 4 $\int \frac{1}{5-4\sin x}dx$

Solution $I = \int \frac{1}{5-4\sin x}dx = \int \frac{1}{5-\frac{8\tan(x/2)}{1+\tan^2(x/2)}}dx = \int \frac{\sec^2(x/2)}{5\tan^2(x/2)-8\tan(x/2)+5}$

Put $\tan\left(\frac{x}{2}\right) = t \Rightarrow \sec^2\left(\frac{x}{2}\right)dx = dt \Rightarrow I = \int \frac{2dt}{5t^2-8t+5}$

$$\frac{2}{5}\int \frac{dt}{t^2-\frac{8}{5}t+1} = \frac{2}{5}\int \frac{1}{\left(t-\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}$$

$$= \frac{2}{5}\times\frac{1}{\left(\frac{3}{5}\right)}\tan^{-1}\left(\frac{t-\frac{4}{5}}{\frac{3}{5}}\right) + c = \frac{2}{3}\tan^{-1}\left(\frac{5\tan(x/2)-4}{3}\right)$$

Integral of the type

- $\int \frac{dx}{a+b\cos^2 x}$

  Example:

  Let $I = \int \frac{dx}{2+3\cos^2 x} \Rightarrow I = \int \frac{dx}{2(\cos^2 x+\sin^2 x)+3\cos^2 x} = \int \frac{dx}{5\cos^2 x+2\sin^2 x}$

  $$= \int \frac{dx}{\cos^2 x(5+2\tan^2 x)} = \int \frac{\sec^2 x dx}{5+2\tan^2 x}$$

  Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\Rightarrow I = \int \frac{dx}{5+2t^2} = \int \frac{dx}{\left(\sqrt{5}\right)^2 + \left(\sqrt{2t}\right)^2} = \frac{1}{\sqrt{10}} \tan^{-1}\left(\sqrt{\frac{2\tan x}{5}}\right) + c = \frac{1}{\sqrt{10}} \tan^{-1}\left(\sqrt{\frac{2\tan x}{5}}\right) + c$$

- $\int \frac{dx}{a + b\sin^2 x}$

Example: Evaluate $\int \frac{dx}{2+3\sin^2 x}$

Solution: Let $I = \int \frac{dx}{2+3\sin^2 x} = \int \frac{dx}{2\sin^2 x + 2\cos^2 x + 3\sin^2 x} = \int \frac{dx}{2\cos^2 x + 5\sin^2 x}$

$$= \int \frac{dx}{\cos^2 x\left(2+5\tan^2 x\right)} = \int \frac{\sec^2 dx}{2+5\tan^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 xdx = dt$

$$I = \int \frac{dt}{2+5t^2} = \int \frac{\mathrm{dt}}{\left(\sqrt{2}\right)^2 + \left(\sqrt{5t}\right)^2} = \frac{1}{\sqrt{10}} \tan^{-1}\left(\sqrt{\frac{5t}{2}}\right) + c = \frac{1}{\sqrt{10}} \tan^{-1}\left(\sqrt{\frac{5\tan x}{2}}\right) + c$$

- $\int \frac{dx}{a + b\cos^2 x + c\cos^2 x}$

Example: Evaluate $\int \frac{dx}{2+\sin^2 x + \cos^2 x}$

Solution: Let $I = \int \frac{dx}{2+\sin^2 x+\cos^2 x} = \int \frac{dx}{3\sin^2 x + 2\cos^2 x}$

This can now be evaluated by the method of previous example.

- $\int \frac{dx}{a\cos^2 x + b\sin^2 x + c\sin x\cos x}$

Evaluate: $\int \frac{dx}{\cos^2 x + 2\sin^2 x + 2\sin x\cos x}$

Solution: Let $I = \int \frac{dx}{2\cos^2 x + \sin^2 x + 2\sin x\cos x} = \int \frac{dx}{\cos^2 x\left(2+\tan^2 x + 2\tan x\right)}$

$$= \int \frac{\sec^2 xdx}{\left(2+\tan^2 x+2\tan x\right)}$$

Put $\tan x = t \Rightarrow \sec^2 xdx = dt$

$$\Rightarrow I = \int \frac{dt}{t^2+2t+2} = \int \frac{dt}{(t+1)^2+1} = \tan^{-1}(t+1) + c = \tan^{-1}(\tan x + 1) + c$$

- $\int \frac{dx}{(a\cos x + b\sin x)^2}$

Method: $\int \frac{dx}{(a\cos x+b\sin x)^2}=\int \frac{dx}{a\cos^2 x+b\sin^2 x+2ab\sin x\cos x}$

This can be evaluated with the help of previous example.

- $\int \frac{\phi(\tan x)\,dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x+d}$

Example: $\int \frac{2\tan x+3}{\sin^2 x+2\cos^2 x}dx$

Solution: Let $I=\int \frac{2\tan x+3}{\sin^2 x+2\cos^2 x}dx=\int \frac{(2\tan x+3)\sec^2 xdx}{\tan^2 x+2}$

Put $\tan x=t \Rightarrow \sec^2 xdx=dt$

$\Rightarrow I=\int \frac{2t+3}{t^2+2}dt=\int \frac{2tdt}{t^2+2}+\int \frac{3}{t^2+\left(\sqrt{2}\right)^2}dt=\log\left|t^2+2\right|+\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+c$

$=\log\left(\tan^2 x+2\right)+\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)+c$

Integrals of the forms:

- $\int \sqrt{\sec^2 x\pm a}dx$
- $\int \sqrt{\cos ec^2 x\pm a}dx$
- $\int \sqrt{\tan^2 x\pm a}dx$
- $\int \sqrt{\cot^2 x\pm a}dx$

Method:

i. Write $\sqrt{\sec^2 x\pm a}=\frac{\sec^2 x\pm a}{\sqrt{\sec^2 x\pm a}}=\frac{\sec^2 x}{\sqrt{\sec^2 x\pm a}}\pm\frac{a\cos x}{\sqrt{1\pm a\cos^2 x}}$

In the first part put $t=\tan x$ and in the second part put $t=\sin x$

ii. Write $\sqrt{\text{cosec}^2 x\pm a}=\frac{\text{cosec}^2 x\pm a}{\sqrt{\text{cosec}^2 x\pm a}}=\frac{\text{cosec}^2 x}{\sqrt{\text{cosec}^2 x\pm a}}\pm\frac{a\sin x}{\sqrt{1\pm a\sin^2 x}}$

In the first part put $t=\cot x$ and in the second part put $t=\cos x$

iii. In case of $\sqrt{\tan^2 x\pm a}$ or $\sqrt{\cot^2 x\pm a}$ change $\tan^2 x$ into $\sec^2 x-1$ & $\cot^2 x$ into $\text{cosec}^2 x-1$

Example: Evaluate $\int \sqrt{\sec^2 x+1}dx$

Solution: Let $I=\int \sqrt{\sec^2 x+1}dx=\int \frac{\sec^2 x+1}{\sqrt{\sec^2 x+1}}dx=\int \frac{\sec^2 x}{\sqrt{\sec^2 x+1}}dx+\int \frac{\cos xdx}{\sqrt{1+\cos^2 x}}$

$=\int \frac{\sec^2 xdx}{\sqrt{2+\tan^2 x}}+\int \frac{\cos xdx}{\sqrt{2-\sin^2 x}}$

For the first part put $t=\tan x\Rightarrow dt=\sec^2 xdx$ & for the second part put $y=\sin x\Rightarrow dy=\cos xdx$

$$\Rightarrow I=\int\frac{dt}{\sqrt{2+t^2}}+\int\frac{dy}{\sqrt{2-y^2}}=\log\left|t+\sqrt{2+t^2}\right|+\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)+c$$

$$=\log\left|\tan x+\sqrt{2+\tan^2 x}\right|+\sin^{-1}\left(\frac{\sin x}{\sqrt{2}}\right)+c$$

Integral of the type $\int\frac{dx}{a\cos x+b\sin x}$

Method: Put $a=r\cos\alpha$ & $b=r\sin\alpha$ where $r=\sqrt{a^2+b^2}$ & $\alpha=\tan^{-1}\left(\frac{b}{a}\right)$

$$\Rightarrow I=\int\frac{dx}{r\cos x\cos\alpha+r\sin x\sin\alpha}=\frac{1}{\sqrt{a^2+b^2}}\int\frac{dx}{\cos(x-\alpha)}=\frac{1}{\sqrt{a^2+b^2}}\int\sec(x-\alpha)\,dx$$

$$\Rightarrow I=\frac{1}{\sqrt{a^2+b^2}}\log\left|\sec\left(x-\tan^{-1}\frac{b}{a}\right)+\tan\left(x-\tan^{-1}\frac{b}{a}\right)\right|+c$$

Example: $\int\frac{dx}{2\cos x+3\sin x}$

Solution: Let $I=\int\frac{dx}{2\cos x+3\sin x}$

Put $2=r\cos\alpha$ & $3=r\sin\alpha\Rightarrow r=\sqrt{13}$ & $\alpha=\tan^{-1}\left(\frac{3}{2}\right)$

$$\Rightarrow I=\frac{1}{\sqrt{13}}\int\frac{dx}{\cos\alpha\cos x+\sin\alpha\sin x}=\frac{1}{\sqrt{13}}\int\frac{dx}{\cos(x-\alpha)}$$

$$=\frac{1}{\sqrt{13}}\int\sec(x-\alpha)\,dx=\frac{1}{\sqrt{13}}\log\left|\sec\left(x-\tan^{-1}\frac{3}{2}\right)+\tan\left(x-\tan^{-1}\frac{3}{2}\right)\right|+c$$

Integral of the type $\int\frac{a\cos x+b\sin x}{c\cos x+d\sin x}dx$

Method: Write $a\cos x+b\sin x=\lambda(c\cos x+d\sin x)+\mu\frac{d}{dx}(c\cos x+d\sin x)$

$$i.e. Num=\lambda(Deno)+\mu\frac{d(Deno)}{dx}$$

$$I=\int\frac{\lambda(c\cos x+d\sin x)+\mu(-c\sin x+d\cos x)}{c\cos x+d\sin x}dx=\lambda\int dx+\mu\int\frac{-c\sin x+d\cos x}{c\cos x+d\sin x}dx$$

$$=\lambda x+\mu\log\left|c\cos x+d\sin x\right|+K$$

Example: $\int\frac{3\sin x+2\cos x}{3\cos x+2\sin x}dx$

Solution: Write $3\sin x+2\cos x=\lambda(3\cos x+2\sin x)+\mu(-3\sin x+2\cos x)$

Comparing the coefficients of $\sin x$ & $\cos x$ we get the following equations

$$3=2\lambda-3\mu\ \&\ 2=3\lambda+2\mu\Rightarrow\lambda=\frac{12}{13}\ \&\ \mu=-\frac{5}{13}$$

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$$\Rightarrow I=\lambda\int dx+\mu\int\frac{-3\sin x+2\cos x}{3\cos x+2\sin x}dx=\frac{12}{13}x-\frac{5}{13}\log\left|3\cos x+2\sin x\right|+c$$

Integral of the type $\int\frac{p\cos+q\sin x+r}{a\cos x+b\sin x+c}dx$

Method: Write $Num=\lambda(Deno)+\mu\frac{d}{dx}(Deno)+\gamma$

$$\Rightarrow I=\lambda\int dx+\mu\int\frac{d.c.\text{ of denominator}}{\text{denominator}}dx+\gamma\int\frac{1}{a\cos x+b\sin x+c}dx$$

Example: $\int\frac{2+3\cos x}{\sin x+2\cos x+3}dx$

Solution: Write $2+3\cos x=\lambda(\sin x+2\cos x+3)+\mu(\cos x-2\sin x)+\gamma$

Comparing the coefficients of $\sin x,\cos x$ and constant terms we get

$$\lambda-2\mu=0,2\lambda+\mu=3\ \&\ 3\lambda+\gamma=2\Rightarrow\lambda=\frac{6}{5},\mu=\frac{3}{5}\ \&\ \gamma=\frac{-8}{5}$$

$$\Rightarrow I=\frac{6}{5}\int 1.dx+\frac{3}{5}\int\frac{\cos x-2\sin x}{\sin x+2\cos x+3}dx-\frac{8}{5}\int\frac{1}{\sin x+2\cos x+3}$$

$$=\frac{6}{5}x+\frac{3}{5}\log\left|\sin x+2\cos x+3\right|-\frac{8}{5}I_3$$

where $I=\int\frac{dx}{\sin x+2\cos x+3}=\int\frac{dx}{\frac{2\tan(x/2)}{1+\tan^2(x/2)}+2\left(\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right)+3}$

$$=\int\frac{\sec^2(x/2)dx}{\tan^2(x/2)+2\tan(x/2)+5}$$

Let $\tan(x/2)=t\Rightarrow\frac{1}{2}\sec^2(x/2)dx=dt\Rightarrow\sec^2(x/2)dx=2dt$

$$I=\int\frac{2dt}{t^2+2t+5}=2\int\frac{dt}{(t+1)^2+(2)^2}=2\cdot\frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right)+c=\tan^{-1}\left(\frac{\tan(x/2)+1}{2}\right)+c$$

Thus $I==\frac{6}{5}x+\frac{3}{5}\log\left|\sin x+2\cos x+3\right|-\frac{8}{5}\tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)+1}{2}\right)+c$

Some special integrals:

1. $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$
2. $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a-x}{a+x}\right|+c$
3. $\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$

4. $\int \frac{dx}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a} + c$

5. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$

6. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

7. $\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

8. $\int \sqrt{x^2 - a^2}\,dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

9. $\int \sqrt{x^2 + a^2}\,dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + c$

Application of these formulae: The above standard integrals are very important. We give below the integrals which are application of these types:

Type A:

(i) $\int \frac{dx}{ax^2 + bx + c}$ (ii) $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ (iii) $\int \sqrt{ax^2 + bx + c}\,dx$

Write $ax^2 + bx + c = a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right\}$

Example: $\int \frac{dx}{2x^2 + x - 1}$

Solution: $\int \frac{dx}{2x^2 + x - 1} = \frac{1}{2}\int \frac{dx}{x^2 + \frac{1}{2}x - \frac{1}{2}} = \frac{1}{2}\int \frac{dx}{\left(x + \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2}$

$$= \frac{1}{2} \times \frac{1}{2\left(\frac{3}{4}\right)}\log\left|\frac{x + \frac{1}{4} - \frac{3}{4}}{x + \frac{1}{4} + \frac{3}{4}}\right| + c = \frac{1}{3}\log\left|\frac{4x - 2}{4x + 4}\right| + c = \frac{1}{3}\log\left|\frac{2x - 1}{2x + 2}\right| + c$$

Example: $\int \frac{dx}{\sqrt{2x^2 + x - 1}}$

Solution: $\int \frac{dx}{\sqrt{2x^2 + x - 1}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2 + \frac{1}{2}x - \frac{1}{2}}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(x + \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2}}$

$$= \frac{1}{\sqrt{2}}\log\left|x + \frac{1}{4} + \sqrt{\left(x + \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2}\right| + c$$

Example: $\int \sqrt{x^2 + x + 1}\,dx$

Solution: $\int \sqrt{x^2+x+1}\,dx = \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$

$$= \frac{1}{2}\left(x+\frac{1}{2}\right)^2 \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{3}{8}\log\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right| + c$$

Example: $\int \frac{1}{3+2x-x^2}\,dx$

Solution:

$$\int \frac{1}{3+2x-x^2}\,dx = \int \frac{1}{-\left(x^2-2x-3\right)}\,dx = \int \frac{1}{-\left\{(x-1)^2-2^2\right\}}\,dx = \int \frac{1}{2^2-(x-1)^2}\,dx$$

$$= \frac{1}{2(2)}\log\left|\frac{2+(x-1)}{2-(x-1)}\right| + c = \frac{1}{4}\log\left|\frac{1+x}{3-x}\right| + c$$

Example: $\int \sqrt{3+2x-x^2}\,dx$

Solution: $\int \sqrt{3+2x-x^2}\,dx = \int \sqrt{-\left(x^2-2x-3\right)}\,dx = \int \sqrt{-\left\{(x-1)^2-2^2\right\}}\,dx$

$$= \int \sqrt{2^2-(x-1)^2}\,dx = \frac{(x-1)\sqrt{2^2-(x-1)^2}}{2} + 2\sin^{-1}\left(\frac{x-1}{2}\right) + c$$

Example: $\int \frac{dx}{4x^2-4x+3}$

Solution: $\int \frac{dx}{4x^2-4x+3} = \int \frac{1}{4\left(x^2-x+\frac{3}{4}\right)}\,dx = \frac{1}{4}\int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\,dx$

$$= \frac{1}{4}\times\frac{1}{\left(1/\sqrt{2}\right)}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{\sqrt{2}}}\right) + c = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + c$$

Type B:

(i) $\int \frac{px+q}{ax^2+bx+c}\,dx$ (ii) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ (iii) $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

Write $px+q = \lambda\frac{d}{dx}\left(ax^2+bx+c\right) + \mu = \lambda(2ax+b) + \mu$

Example $\int \frac{3x+4}{\sqrt{2x^2+3x+1}}\,dx$

Solution: Let $\int \frac{3x+4}{\sqrt{2x^2+3x+1}}dx$

Write $3x+4=\lambda(4x+3)+\mu \Rightarrow 4\lambda=3 \;\&\; 3\lambda+\mu=4 \Rightarrow \lambda=\frac{3}{4} \;\&\; \mu=\frac{7}{4}$

$$I=\frac{3}{4}\int\frac{4x+3}{\sqrt{2x^2+3x+1}}dx+\frac{7}{4}\int\frac{1}{\sqrt{2x^2+3x+1}}dx=\frac{3}{4}I_1+\frac{7}{4}I_2$$

$I_1=\int\frac{4x+3}{\sqrt{2x^2+3x+1}}$ Put $2x^2+3x+1=t \Rightarrow (4x+3)dx=dt$

$$\Rightarrow I_1=\int\frac{dt}{\sqrt{t}}=\int t^{-1/2}dt=\frac{t^{-\frac{1}{2}+1}}{\frac{-1}{2}+1}=2\sqrt{t}+C_1=2\sqrt{2x^2+3x+1}+C_1$$

$$I_2=\int\frac{1}{\sqrt{2x^2+3x+4}}dx=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{x^2+\frac{3}{2}x+\frac{1}{2}}}=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}}dx$$

$$=\frac{1}{\sqrt{2}}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}\right|+c$$

Thus $I=\frac{3}{2}\sqrt{2x^2+3x+1}+\frac{7}{4\sqrt{2}}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}\right|+c$

Example: $\int\frac{3x+4}{2x^2+3x+1}dx$

Solution: Let $I=\int\frac{3x+4}{2x^2+3x+1}dx$

Write $3x+4=\lambda(4x+3)+\mu \Rightarrow 4\lambda=3 \;\&\; 3\lambda+\mu=4 \Rightarrow \lambda=\frac{3}{4} \;\&\; \mu=\frac{7}{4}$

$$I=\frac{3}{4}\int\frac{4x+3}{2x^2+3x+1}dx+\frac{7}{4}\int\frac{1}{2x^2+3x+1}dx=\frac{3}{4}I_1+\frac{7}{4}I_2$$

Put $2x^2+3x+1=t \Rightarrow (4x+3)dx=dt \Rightarrow I_1=\int\frac{dt}{t}=\log|t|+C_1=\log\left|2x^2+3x+1\right|+C_1$

$$I_2=\frac{1}{2}\int\frac{1}{x^2+\frac{3}{2}x+\frac{1}{3}}dx=\frac{1}{2}\int\frac{1}{\left(x+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}dx$$

$$=\log\left|\frac{x+\frac{3}{4}-\frac{1}{4}}{x+\frac{3}{4}+\frac{1}{4}}\right|+C_2=\log\left|\frac{x+\frac{1}{2}}{x+1}\right|+C_2$$

Thus $I=\frac{3}{4}\log\left|2x^2+3x+1\right|+\frac{7}{4}\log\left|\frac{x+\frac{1}{2}}{x+1}\right|+C$

Example: $\int(x+1)\sqrt{1-x-x^2}\,dx$

Solution: Put $x+1=\lambda(-1-2x)+\mu\Rightarrow\lambda=-\frac{1}{2}\ \&\ \mu=\frac{1}{2}$

Thus $I=\int(x+1)\sqrt{1-x-x^2}\,dx=\frac{-1}{2}\int(-2x-1)\sqrt{1-x-x^2}\,dx+\frac{1}{2}\int\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}$

$=\frac{-1}{2}\int\sqrt{t}\,dt+\frac{1}{2}\int\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}$ $\left(\text{Put } 1-x-x^2=t\Rightarrow(-2x-1)\,dx=dt\right)$

$$-\frac{1}{3}\left(1-x-x^2\right)^{\frac{3}{2}}+\frac{\left(x+\frac{1}{2}\right)\sqrt{\left(\frac{5}{\sqrt{2}}\right)^2-\left(x+\frac{1}{2}\right)^2}}{4}+\frac{5}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)+c$$

Type C:

$(i)\int\frac{px^2+qx+r}{ax^2+bx+c}dx\quad(ii)\int\frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}dx\quad(iii)\int\left(px^2+qx+r\right)\sqrt{ax^2+bx+c}\,dx$

Method: Write $px^2+qx+r=\lambda\left(ax^2+bx+c\right)+\mu(2ax+b)+\gamma$

Integral in (i) will become

$$I=\int\frac{\lambda(ax^2+bx+c)+\mu(2ax+b)+\gamma}{ax^2+bx+c}dx=\lambda\int dx+\mu\int\frac{2ax+b}{ax^2+bx+c}dx+\gamma\int\frac{1}{ax^2+bx+c}dx$$

Example: $\int\frac{2x^2+5x+4}{\sqrt{x^2+x+1}}dx$

Solution: Let $I=\int\frac{2x^2+5x+4}{\sqrt{x^2+x+1}}dx$

Put $2x^2+5x+4=\lambda\left(x^2+x+1\right)+\mu\frac{d}{dx}\left(x^2+x+1\right)+\gamma$

Or $2x^2+5x+4=\lambda\left(x^2+x+1\right)+\mu(2x+1)+\gamma$

On comparing coefficients we get $\lambda=2,\mu=\frac{3}{2}\ \&\ \gamma=\frac{1}{2}$

$$\Rightarrow I=\int \frac{2\left(x^2+x+1\right)+\frac{3}{2}(2x+1)+\frac{1}{2}}{\sqrt{x^2+x+1}}dx=2\int \sqrt{x^2+x+1}\,dx+\frac{3}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}dx+\frac{1}{2}\int \frac{1}{\sqrt{x^2+x+1}}dx$$

$$=2\int \sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx+\frac{1}{2}\int \frac{dt}{\sqrt{t}}+\frac{3}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}dx \text{ where}\left(t=x^2+x+1\right)$$

$$=2\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{1}{2}\cdot\frac{3}{4}\log\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|\right\}$$

$$+\frac{3}{2}\frac{t^{1/2}}{1/2}+\frac{1}{2}\log\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|+c$$

$$=2\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{1}{2}\cdot\frac{3}{4}\log\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|\right\}$$

$$+3\sqrt{x^2+x+1}+\frac{1}{2}\log\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|+c$$

Example: $\int \frac{2x^2+5x+4}{x^2+x+1}dx$

Solution: Let $\int \frac{2x^2+5x+4}{x^2+x+1}dx$

Put $2x^2+5x+4=\lambda\left(x^2+x+1\right)+\mu\frac{d}{dx}\left(x^2+x+1\right)+\gamma$

Or $2x^2+5x+4=\lambda\left(x^2+x+1\right)+\mu(2x+1)+\gamma$

On comparing coefficients we get $\lambda=2,\mu=\frac{3}{2}\ \&\ \gamma=\frac{1}{2}$

$$\Rightarrow I=\int \frac{2\left(x^2+x+1\right)+\frac{3}{2}(2x+1)+\frac{1}{2}}{x^2+x+1}dx=2\int dx+\frac{3}{2}\int \frac{2x+1}{x^2+x+1}dx+\frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

$$=2\int dx+\frac{1}{2}\int \frac{dt}{t}+\frac{3}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx \text{ where}\left(t=x^2+x+1\right)$$

$$2x+\frac{3}{2}\log|t|+\frac{1}{2}\times\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c=2x+\frac{3}{2}\log\left|x^2+x+1\right|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

Integration of the type: $\int f\left(x,\left(\frac{ax+b}{cx+d}\right)^{\frac{\alpha}{n}}\right)\frac{1}{x}dx$

Method: Put $\frac{ax+b}{cx+d}=t^n$

Example: Evaluate $\int\left(\frac{x+2}{2x+3}\right)^{\frac{1}{2}}\frac{1}{x}dx$

Solution: Put $\frac{x+2}{2x+3}=t^2\Rightarrow x=\frac{3t^2-2}{1-2t^2}\Rightarrow dx=\frac{-2tdt}{\left(1-2t^2\right)^2}$

$$\Rightarrow I=-2\int t\cdot\left(\frac{1-2t^2}{3t^2-2}\right)\left(\frac{-2tdt}{\left(1-2t^2\right)^2}\right)=4\int\frac{t^2}{\left(3t^2-2\right)\left(1-2t^2\right)}dt$$

Let $t^2=y\Rightarrow\frac{t^2}{\left(3t^2-2\right)\left(1-2t^2\right)}=\frac{y}{(3y-2)(1-2y)}=\frac{A}{3y-2}+\frac{B}{1-2y}$

$\Rightarrow A(1-2y)+B(3y-2)=y$. Put $y=\frac{1}{2}\Rightarrow B=-1$.Put $y=\frac{2}{3}\Rightarrow A=-2$

Thus $I=\int\frac{-2}{3t^2-2}dt+\int\frac{-1}{\left(1-2t^2\right)}dt=-\frac{2}{3}\int\frac{dt}{t^2-\left(\sqrt{2/3}\right)^2}-\frac{1}{2}\int\frac{dt}{\left(\sqrt{1/2}\right)^2-t^2}$

These integrals can be evaluated with the help of the formulae of some special integrals.

Integral of the type: $\int f\left(x,(ax+b)^{\alpha/n},(ax+b)^{\beta/m}\right)dx$

Method: Put $ax+b=t^p$ where p is L.C.M of m & n

Example: Evaluate $\int\frac{dx}{\sqrt{x+1}-(x+1)^{1/4}}$

Solution: Here $n=2$ & $m=4\Rightarrow p=LCM$ of 2 & 4 = 4

Thus put $x+1=t^4\Rightarrow dx=4t^3dt$

$$\Rightarrow I=\int\frac{4t^3dt}{t^2-t}=\int\frac{4t^2dt}{t-1}=4\int\frac{t^2-1+1}{t-1}dt=4\int\frac{t^2-1}{t-1}dt+4\int\frac{dt}{t-1}=4\int(t+1)dt+4\int\frac{dt}{t-1}$$

$$=4\cdot\frac{t^2}{2}+4t+4\log|t-1|+C=2\sqrt{x+1}+4\sqrt[4]{x+1}+4\log\left|\sqrt[4]{x+1}-1\right|+C$$

Integral of the type: $\int\frac{1}{(ax+b)\sqrt{px+q}}dx$

Method: Put $px+q=t^2$

Example: Evaluate $\int \frac{dx}{(x+3)\sqrt{x+2}}$

Solution: Put $x+2=t^2 \Rightarrow x=t^2-2 \Rightarrow dx=2tdt \Rightarrow I=\int \frac{2tdt}{(t^2+1)t}=2\int \frac{dt}{t^2+1}$

$= 2\tan^{-1}(t)+C = 2\tan^{-1}\left(\sqrt{x+2}\right)+C$

Integral of the type: $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$

Method: Put $px+q=t^2$

Example: Evaluate $\int \frac{dx}{(x^2+2x+2)\sqrt{x+1}}$

Solution: Put $x+1=t^2 \Rightarrow x=t^2-1 \Rightarrow dx=2tdt$

$$\Rightarrow I=\int \frac{2tdt}{\left\{(t^2-1)^2+2(t^2-1)+2\right\}t}=2\int \frac{dt}{t^4+1}$$

This integral will be discussed later.

Integral of the type: $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$

Method: Put $px+q=\frac{1}{t}$

Example: Evaluate $\int \frac{dx}{(x+1)\sqrt{x^2+2x+2}}$

Solution: Put $x+1=\frac{1}{t} \Rightarrow dx=-\frac{1}{t^2}dt$

$$\Rightarrow I=\int \frac{\frac{-1}{t^2}dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2+2\left(\frac{1}{t}-1\right)+2}}=\int \frac{\frac{-1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}+1}}=\int \frac{-dt}{\sqrt{t^2+1}}=-\log\left|t+\sqrt{t^2+1}\right|+C$$

$$=-\log\left|\frac{1}{x+1}+\sqrt{1+\frac{1}{(x+1)^2}}\right|+C$$

Integral of the type: $\int \frac{dx}{Q_1(x)\sqrt{Q_2(x)}}$ where $Q_1(x)$ & $Q_2(x)$ are two quadratic polynomials.

Method: Put $x=\frac{1}{t}$ or $\frac{Q_2(x)}{Q_1(x)}=t^2$

Example: Evaluate $\int \frac{dx}{(x^2+1)\sqrt{x^2+2}}$

Solution: Put $x=\frac{1}{t}\Rightarrow dx=-\frac{1}{t^2}dt\Rightarrow I=\int\frac{-\frac{1}{t^2}dt}{\left(\frac{1}{t^2}+1\right)\sqrt{\frac{1}{t^2}+2}}=\int\frac{-tdt}{\left(1+t^2\right)\sqrt{1+2t^2}}$

Put $1+2t^2=y^2\Rightarrow 4tdt=2ydy\Rightarrow I=\frac{-1}{2}\int\frac{ydy}{\left(\frac{y^2-1}{2}\right)y}=-\int\frac{dy}{y^2-1}$

$$I=\frac{-1}{2}\log\left|\frac{y-1}{y+1}\right|+C=\frac{-1}{2}\log\left|\frac{\sqrt{1+2t^2}-1}{\sqrt{1+2t^2}+1}\right|+C=-\frac{1}{2}\log\left|\frac{\sqrt{1+\frac{2}{x^2}}-1}{\sqrt{1+\frac{2}{x^2}}+1}\right|+C=\frac{-1}{2}\log\left|\frac{\sqrt{x^2+2}-x}{\sqrt{x^2+2}+x}\right|+C$$

Integral of the type: $\int f\left\{\left(x\pm\sqrt{x^2+a^2}\right)^n\right\}dx$

Method: Put $x\pm\sqrt{x^2+a^2}=t$

Example: Evaluate $\int\left(x+\sqrt{a^2+x^2}\right)^2 dx$

Solution:

Put $x+\sqrt{a^2+x^2}=t\Rightarrow\sqrt{a^2+x^2}=t-x\Rightarrow a^2+x^2=t^2+x^2-2tx\Rightarrow x=-\left(\frac{a^2-t^2}{2t}\right)$

$$\Rightarrow dx=\left(\frac{1}{2}+\frac{a^2}{2t^2}\right)dt\Rightarrow I=\int\left(\frac{t^2}{2}+\frac{a^2}{2}\right)dt=\frac{t^3}{6}+\frac{a^2}{2}t+C$$

$$=\frac{\left(x+\sqrt{a^2+x^2}\right)^3}{6}+\frac{a^2}{2}\left(x+\sqrt{a^2+x^2}\right)+C$$

Integration of rational functions using Partial Fractions:

Some Basic Definitions:

(a) **Polynomial of degree *n*:** An expression of the type $P(x)=a_0x^n+a_1x^{n-1}+a_2x^2+\cdots+a_{n-1}x+a_n$ where $a_0,a_1,a_2\cdots,a_n$ are real numbers, $a_0\neq 0$ and n is positive integer is called a polynomial of degree n.

(b) **Rational Function:** A function of the form $\frac{P}{Q}$ where P & Q are polynomials is called a rational function. Consider the rational function $\frac{x+7}{(2x-3)(3x+4)}=\frac{1}{2x-3}-\frac{1}{3x+4}$. The two fractions on RHS are called **Partial Fractions.**

(c) **Proper and Improper Fractions:** Any rational algebraic function is called a p[roper fraction if the degree of numerator is less than that of its denominator, otherwise it is called an improper fraction.

For example $\frac{x^2+1}{x^3+x+1}$ is a proper fraction, where as $\frac{x^5-x+1}{x^2+3x+9}$ is improper fraction.

Note: In using the method of partial fractions for $\frac{P(x)}{Q(x)}$, we must have $Deg\left(P(x)\right) < Deg\left(Q(x)\right)$. If it is not so, we carry out the division of $P(x)$ by $Q(x)$ and reduce the degree of numerator to less than the degree of denominator.

$i.e. \frac{P(x)}{Q(x)} = P_1(x) + \frac{P_2(x)}{Q(x)}$ where $Deg\left(P_2(x)\right) < Deg\left(Q(x)\right)$

The partial fractions depend on the nature of $Q(x)$. We have to deal with the following type different types when the factors of $Q(x)$ are:

1. **Linear and Non- Repeated**
2. **Linear and Repeated**
3. **Quadratic and non-repeated**
4. **Quadratic and repeated.**

Case1. When denominator is expressible as the product of non-repeated linear factors:

Let $Q(x) = (x-a_1)(x-a_2)(x-a_3)\cdots(x-a_n)$

Then we assume that;

$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \frac{A_3}{(x-a_3)} + \cdots + \frac{A_n}{(x-a_n)}$ where $A_1, A_2, A_3, \cdots, A_n$ are constants and can be determined by equating numerator on LHS to numerator on RHS and then substituting $x = a_1, a_2, \cdots, a_n$

Example: Evaluate $\int \frac{1}{(x-1)(x+2)(2x+3)} dx$

Solution: Let $\frac{1}{(x-1)(x+2)(2x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(2x+3)}$

$\Rightarrow A(x+2)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+2) = 1$

Put $x-1=0$ or $x=1$ we get $A(1+2)(2+3) + B(0) + C(0) = 1 \Rightarrow A = \frac{1}{15}$

Similarly for getting B, let $x+2=0$ or $x=-2$, we get $A(0) + B(-2-1)(-4+3) + C(0) = 1 \Rightarrow B = \frac{1}{3}$

For getting C, let $2x+3=0 \Rightarrow x = -\frac{3}{2}$, we get

$A(0) + B(0) + C\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}+2\right) = 1 \Rightarrow C = -\frac{4}{5}$

Hence, $\frac{1}{(x-1)(x+2)(2x+3)} = \frac{1}{15(x-1)} + \frac{1}{3(x+2)} - \frac{4}{5(2x+3)}$

$$\Rightarrow \int \frac{1}{(x-1)(x+2)(2x+3)} dx = \frac{1}{15}\int \frac{1}{x-1} dx + \frac{1}{3}\int \frac{1}{x+2} dx - \frac{4}{5}\int \frac{1}{2x+3} dx$$

$$\Rightarrow \int \frac{1}{(x-1)(x+2)(2x+3)} dx = \frac{1}{15}\log|x-1| + \frac{1}{3}\log|x+2| - \frac{4}{5}\times\frac{1}{2}\log|2x+3| + C$$

Example: Evaluate $\int \frac{x+1}{(x-1)(x-2)(x-3)} dx$

Solution: Let $\frac{x+1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$\Rightarrow A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) = x+1$

Put $x = 1 \Rightarrow A(1-2)(1-3) = 1+1 \Rightarrow A = 1$

Put $x = 2 \Rightarrow B(2-1)(2-3) = 2+1 \Rightarrow B = -1$

Put $x = 3 \Rightarrow C(3-1)(3-2) = 3+1 \Rightarrow C = 2$

$$\Rightarrow \int \frac{x+1}{(x-1)(x-2)(x-3)} dx = \int \frac{1}{x-1} dx - \int \frac{1}{x-2} dx + 2\int \frac{1}{x-3} dx$$

$= \log|x-1| - \log|x-2| + 2\log|x-3| + C$

Example: Evaluate $\int \frac{3x^3 + 2x^2 + x + 1}{(x+1)(x+2)} dx$

Solution: $\frac{3x^3 + 2x^2 + x + 1}{(x+1)(x+2)} = Ax + B + \frac{C}{x+1} + \frac{D}{x+2}$

$\Rightarrow Ax(x+1)(x+2) + B(x+1)(x+2) + C(x+2) + D(x+1) = 3x^3 + 2x^2 + x + 1$

Put $x = -1 \Rightarrow C = -1$

Put $x = -2 \Rightarrow D = 17$

Compare coefficients of x^3 we get $A = 3$

Compare coefficients of x^2 we get $3A + B = 2 \Rightarrow B = 2 - 3A = 2 - 9 = -7$

Thus we get $\frac{3x^3 + 2x^2 + x + 1}{(x+1)(x+2)} = 3x - 7 - \frac{1}{x+1} + \frac{17}{x+2}$

$$\int \frac{3x^3 + 2x^2 + x + 1}{(x+1)(x+2)} dx = \int (3x-7) dx - \int \frac{1}{x+1} dx + 17\int \frac{1}{x+2} dx$$

$$\int \frac{3x^3 + 2x^2 + x + 1}{(x+1)(x+2)} dx = \frac{3x^2}{2} - 7x - \log|x+1| + 17\log|x+2| + C$$

Case2: When the denominator $Q(x)$ is expressible as the product of the linear factors such that some of them are repeating. (**Linear and Repeated**)

Let $Q(x)=(x-a)^k(x-a_1)(x-a_2)(x-a_3)\cdots(x-a_r)$. Then we assume that

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_k}{(x-a)^k}+\frac{B_1}{(x-a_1)}+\frac{B_2}{(x-a_2)}+\cdots+\frac{B_r}{(x-a_r)}$$

Example: Evaluate $\int\frac{3x-2}{(x-1)^2(x+1)(x+2)}dx$

Solution: Let $\frac{3x-2}{(x-1)^2(x+1)(x+2)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}+\frac{D}{(x+2)}$

$$\Rightarrow A(x-1)(x+1)(x+2)+B(x+1)(x+2)+C(x-1)^2(x+2)+D(x-1)^2(x+1)$$

Put $x=1$ we get $B=\frac{1}{6}$

Put $x=-1$ we get $C=-\frac{5}{4}$

Put $x=-2$ we get $D=\frac{8}{9}$

Now equating the coefficient of x on both sides we get $0=A+C+D\Rightarrow A=\frac{13}{36}$

$$\frac{3x-2}{(x-1)^2(x+1)(x+2)}=\frac{13}{36(x-1)}+\frac{1}{6(x-1)^2}-\frac{5}{4(x+1)}+\frac{8}{9(x+2)}$$

$$\int\frac{3x-2}{(x-1)^2(x+1)(x+2)}dx=\frac{13}{36}\int\frac{dx}{(x-1)}+\frac{1}{6}\int\frac{dx}{(x-1)^2}-\frac{5}{4}\int\frac{dx}{(x+1)}+\frac{8}{9}\int\frac{dx}{(x+2)}$$

$$=\frac{13}{36}\log|x-1|-\frac{1}{6(x-1)}-\frac{5}{4}\log|x+1|+\frac{8}{9}\log|x+2|+C$$

Example: Evaluate $\int\frac{x+1}{(x-1)(x+2)^2}dx$

Solution: Let $\frac{x+1}{(x-1)(x+2)^2}=\frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{C}{(x+2)^2}$

$$\Rightarrow A(x+2)^2+B(x-1)(x+2)+C(x-1)=x+1$$

Put $x=1$ we get $A=\frac{2}{9}$

Put $x=-2$ we get $C=\frac{1}{3}$

Comparing coefficient of x^2 we get $A+B=0\Rightarrow B=-A\Rightarrow B=-\frac{2}{9}$

Thus $\frac{x+1}{(x-1)(x+2)^2}=\frac{2}{9(x-1)}-\frac{2}{9(x+2)}+\frac{1}{3(x+2)^2}$

$$\int \frac{x+1}{(x-1)(x+2)^2}dx = \frac{2}{9}\int \frac{dx}{(x-1)} - \frac{2}{9}\int \frac{dx}{(x+2)} + \frac{1}{3}\int \frac{dx}{(x-2)^2}$$

$$\int \frac{x+1}{(x-1)(x+2)^2}dx = \frac{2}{9}\log|x-1| - \frac{2}{9}\log|x+2| - \frac{1}{3(x+2)} + C$$

Case3: When some of the factors in the denominator are non-reducible quadratic but non-repeating

Method: Corresponding to each quadratic factor $ax^2 + bx + c$ we assume the fraction of the type $\frac{Ax+B}{ax^2+bx+c}$.

Example: Evaluate $\int \frac{2x+7}{(x+1)(x^2+4)}dx$

Solution: Let $\frac{2x+7}{(x+1)(x^2+4)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+4)}$

$\Rightarrow A(x^2+4) + (Bx+C)(x+1) = 2x+7$

Put $x = -1 \Rightarrow A = 1$

Comparing coefficients of x^2 we get $A + B = 0 \Rightarrow B = -A \Rightarrow B = -1$

Comparing constant terms we get $4A + C = 7 \Rightarrow C = 7 - 4A = 3$

$$\Rightarrow \frac{2x+7}{(x+1)(x^2+4)} = \frac{1}{x+1} + \frac{(-x+3)}{x^2+4}$$

$$\Rightarrow \int \frac{2x+7}{(x+1)(x^2+4)}dx = \int \frac{1}{x+1}dx + \int \frac{-x+3}{x^2+4}dx = \log|x+1| - \frac{1}{2}\int \frac{2x}{x^2+4}dx + 3\int \frac{dx}{x^2+2^2}$$

$$= \log|x+1| - \frac{1}{2}\log|x^2+4| + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Example: Evaluate $\int \frac{2x+1}{(3x+2)(4x^2+5x+6)}dx$

Solution: Le $\frac{2x+1}{(3x+2)(4x^2+5x+6)} = \frac{A}{3x+2} + \frac{Bx+C}{4x^2+5x+6}$

$\Rightarrow A(4x^2+5x+6) + (Bx+C)(3x+2) = 2x+1$

Put $x = -\frac{2}{3} \Rightarrow A = -\frac{3}{40}$

Comparing coefficients of x^2 and constant terms we get

$4A + 3B = 0 \Rightarrow B = -\frac{4}{3}A \Rightarrow B = \frac{1}{10}$

and $6A + 2C = 1 \Rightarrow C = \frac{1-6A}{2} \Rightarrow C = \frac{29}{40}$

$$\therefore \frac{2x+1}{(3x+2)\left(4x^2+5x+6\right)}=\frac{-3}{40(3x+2)}+\frac{\frac{1}{10}x+\frac{29}{40}}{4x^2+5x+6}=-\frac{3}{40(3x+2)}+\frac{4x+29}{40\left(4x^2+5x+6\right)}$$

$$\Rightarrow \int \frac{2x+1}{(3x+2)\left(4x^2+5x+6\right)}dx=-\frac{3}{40}\int \frac{dx}{3x+2}+\frac{1}{40}\int \frac{4x+29}{4x^2+5x+6}dx$$

$$=-\frac{3}{40\times 3}\log|3x+2|+\frac{1}{40}\times\frac{1}{2}\int \frac{8x+58}{4x^2+5x+6}dx=-\frac{1}{40}\log|3x+2|+\frac{1}{80}\int \frac{8x+5+53}{4x^2+5x+6}dx$$

$$=-\frac{1}{40}\log|3x+2|+\frac{1}{80}\int \frac{8x+5}{4x^2+5x+6}dx+\frac{53}{80}\int \frac{1}{4x^2+5x+6}dx$$

$$=-\frac{1}{40}\log|3x+2|+\frac{1}{80}\log\left|4x^2+5x+6\right|+\frac{53}{320}\int \frac{dx}{x^2+\frac{5}{4}x+\frac{3}{2}}$$

$$=-\frac{1}{40}\log|3x+2|+\frac{1}{80}\log\left|4x^2+5x+6\right|+\frac{53}{320}\int \frac{dx}{\left(x+\frac{5}{8}\right)^2+\left(\frac{\sqrt{71}}{8}\right)^2}$$

$$\int \frac{2x+1}{(3x+2)\left(4x^2+5x+6\right)}dx$$

$$=-\frac{1}{40}\log|3x+2|+\frac{1}{80}\log\left|4x^2+5x+6\right|+\frac{53}{40\sqrt{71}}\tan^{-1}\left(\frac{8x+5}{\sqrt{71}}\right)+C$$

Special Forms of Partial Fractions:

Integral of the type: $\int \frac{dx}{x\left(x^n+k\right)}$

Method: Multiply the numerator and denominator of the integrand by x^{n-1}

$$\Rightarrow \int \frac{dx}{x\left(x^n+k\right)}=\int \frac{x^{n-1}}{x^n\left(x^n+k\right)}dx$$

Now put $x^n=t \Rightarrow nx^{n-1}dx=dt \Rightarrow x^{n-1}dx=\frac{1}{n}dt$

$$\Rightarrow \int \frac{dx}{x\left(x^n+k\right)}=\frac{1}{n}\int \frac{dt}{t(t+k)}=\frac{1}{n}\left\{\int \frac{1}{k}\left(\frac{1}{t}-\frac{1}{t+k}\right)dt\right\}$$

$$=\frac{1}{nk}\{\log|t|-\log|t+k|\}+C=\frac{1}{nk}\log\left|\frac{t}{t+k}\right|+C=\frac{1}{nk}\log\left|\frac{x^n}{x^n+k}\right|+C$$

Example: Evaluate $\int \frac{dx}{x\left(x^5+1\right)}$

Solution: Multiply the numerator and denominator of the integrand by x^4

$$\Rightarrow \int \frac{dx}{x\left(x^5+1\right)}=\int \frac{x^4dx}{x^5\left(x^5+1\right)}$$

Put $x^5 = t \Rightarrow 5x^4 dx = dt \Rightarrow x^4 dx = \frac{1}{5} dt$

$$\Rightarrow \int \frac{dx}{x\left(x^5+1\right)} = \frac{1}{5}\int \frac{dt}{t(t+1)} = \frac{1}{5}\int\left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \frac{1}{5}\left\{\log|t| - \log|t+1|\right\} + C$$

$$= \frac{1}{5}\log\left|\frac{t}{t+1}\right| + C = \frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right| + C$$

Integral containing only even powers of x:

Example: Evaluate $\int \frac{x^2+1}{\left(x^2+2\right)\left(x^2+3\right)\left(x^2+4\right)}dx$

Solution: Put $x^2 = y$ only for partial fraction $\Rightarrow \frac{x^2+1}{\left(x^2+2\right)\left(x^2+3\right)\left(x^2+4\right)} = \frac{y+1}{(y+2)(y+3)(y+4)} = \frac{A}{y+2} + \frac{B}{y+3} + \frac{C}{y+4}$

$$\Rightarrow A(y+3)(y+4) + B(y+2)(y+4) + C(y+2)(y+3) = y+1$$

Put $y = -2 \Rightarrow A = -\frac{1}{2}$

Put $y = -3 \Rightarrow B = 2$

Put $y = -4 \Rightarrow C = -\frac{3}{2}$

$$\frac{x^2+1}{\left(x^2+2\right)\left(x^2+3\right)\left(x^2+4\right)} = -\frac{1}{2(y+2)} + \frac{2}{y+3} - \frac{3}{2(y+4)} = -\frac{1}{2\left(x^2+2\right)} + \frac{2}{\left(x^2+3\right)} - \frac{3}{2\left(x^2+4\right)}$$

$$\int \frac{x^2+1}{\left(x^2+2\right)\left(x^2+3\right)\left(x^2+4\right)} = -\frac{1}{2}\int \frac{dx}{x^2+\sqrt{2}^2} + 2\int \frac{dx}{x^2+\sqrt{3}^2} - \frac{3}{2}\int \frac{dx}{x^2+2^2}$$

$$= -\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{4}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Integral of the type: $\int \frac{dx}{\sin x \times f(\cos x)}$

Method: Multiply the numerator and denominator of the integrand by $\sin x$

$$\Rightarrow I = \int \frac{\sin x dx}{\sin^2 x\left(f(\cos x)\right)} = \int \frac{\sin x dx}{\left(1-\cos^2 x\right)\left(f(\cos x)\right)}$$

Put $\cos x = t \Rightarrow \sin x dx = -dt \Rightarrow I = -\int \frac{dt}{\left(1-t^2\right)\left(f(t)\right)}$

Example: Evaluate $\int \frac{dx}{\sin x(\cos x+2)}$

Solution: $\int \frac{dx}{\sin x(\cos x+2)} = \int \frac{\sin x dx}{\sin^2 x(\cos x+2)} = \int \frac{\sin x dx}{\left(1-\cos^2 x\right)(\cos x+2)}$

Put $\cos x = t \Rightarrow \sin x dx = -dt$

$$\Rightarrow \int \frac{dx}{\sin x(\cos x+2)} = -\int \frac{dt}{(1-t^2)(t+2)} = -\int \frac{dt}{(1-t)(1+t)(t+2)}$$

$$-\frac{1}{6}\int \frac{dt}{1-t} - \frac{1}{2}\int \frac{dt}{1+t} + \frac{1}{3}\int \frac{dt}{t+2}$$

$$= \frac{1}{6}\log|1-t| - \frac{1}{2}\log|1+t| + \frac{1}{3}\log|t+2| + C = \frac{1}{6}\log|1-\cos x| - \frac{1}{2}\log|1+\cos x| + \frac{1}{3}\log|\cos x+2| + C$$