INVERSE TRIGONOMETRIC FUNCTIONS

Def 1: $\sin^{-1}:[-1,1]\rightarrow\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ given by $\sin^{-1}(x)=y\Leftrightarrow\sin y=x$.

Def 2: $\cos^{-1}:[-1,1]\rightarrow[0,\pi]$ given by $\cos^{-1}x=y\Leftrightarrow\cos y=x$

Def 3: $\tan^{-1}:(-\infty,\infty)\rightarrow\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ given by $\tan^{-1}x=y\Leftrightarrow\tan y=x$

Def 4: $\cot^{-1}(-\infty,\infty)\rightarrow\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ given by $\cot^{-1}x=y\Leftrightarrow\cot y=x$

Def 5: $sec^{-1}:(-\infty,1]\rightarrow\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ given by $sec^{-1}x=y\Leftrightarrow sec\,y=x$

Def 6: $cosec^{-1}(-\infty,1]\cup[1,\infty)\rightarrow\left[\frac{-\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ given by $cosec^{-1}x=y\Leftrightarrow cosec\,y=x$

Results

1. $\sin(\sin^{-1}x)=x$ if $-1\le x\le 1$ and $\sin^{-1}(\sin x)=x$ *if* $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$
2. $\cos(\cos^{-1}x)=x$ *if* $-1\le x\le 1$ and $\cos^{-1}(\cos x)=x$ if $0\le x\le\pi$
3. $\tan(\tan^{-1}x)=x\ \forall x\in R$ and $\tan^{-1}(\tan x)=x$ if $-\frac{\pi}{2}<x<\frac{\pi}{2}$
4. $\cot(\cot^{-1}x)=x$ if $x\in R$ and $\cot^{-1}(\cot x)=x$ if $0<x<\pi$
5. $sec(sec^{-1}x)=x$ if $x\in(-\infty,-1]\cup[1,\infty)$ and $sec^{-1}(sec\,x)=x$ if $x\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2,}\pi\right]$
6. $cosec(cosec^{-1}x)=x$ if $x\in(-\infty,-1]\cup[1,\infty)$ and
   $cosec^{-1}(cosec\,x)=x$ if $x\in\left[\frac{-\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$

Results

1. $\sin^{-1}(-x)=-\sin^{-1}x\ \forall x\in[-1,1]$
2. $\cos^{-1}x=\pi-\cos^{-1}x\ \forall x\in[-1,1]$
3. $\tan^{-1}(-x)=-\tan^{-1}x\ \forall x\in R$
4. $\cot^{-1}(-x)=\pi-\cot^{-1}x\ \forall x\in R$
5. $sec^{-1}(-x)=\pi-sec^{-1}x\ \forall x\in(-\infty,-1]\cup[1,\infty)$
6. $cosec^{-1}(-x)=-cosec^{-1}x\ \forall x\in(-\infty,-1]\cup[1,\infty)$

Results

1. $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $-1\le x\le 1$

2. $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$ for any $x\in R$

3. $cosec^{-1}x+sce^{-1}x=\frac{\pi}{2}$ for $-\infty<x\leq-1$ or $1\leq x<\infty$

Results

1. $\sin^{-1}\left(\frac{1}{x}\right)=cosec^{-1}x,\forall x\in\left(-\infty,-1\right]\cup\left[1,\infty\right)$

2. $\cos^{-1}\left(\frac{1}{x}\right)=sec^{-1}x,\forall x\in\left(-\infty,-1\right]\cup\left[1,\infty\right)$

3. $\tan^{-1}\left(\frac{1}{x}\right)=\begin{Bmatrix}\cot^{-1}x \text{ if } x<0\\ -\pi+\cot^{-1}x \text{ for } x<0\end{Bmatrix}$

Results

1. $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$ if $xy<1$

2. $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}$ if $xy>-1$

3. $\sin^{-1}x+\sin^{-1}y=\sin^{-1x}\left\{\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$ if $-1<x,y\leq1$ and $x^2+y^2\leq1$

4. $\sin^{-1}x-\sin^{-1}y=\sin^{-1}x\left\{\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}$ if $-1\leq x,y\leq1$ and $x^2+y^2\leq1$

5. $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left\{xy-\sqrt{1-x^2 1-x^2}\right\}$ if $-1\leq x,y\leq1$ and $x+y\geq0$

6. $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left\{xy+\sqrt{1-x^2 1-x^2}\right\}$ if $-1\leq x,y\leq1$ and $x\leq y$

Example 1: Find the principal value of of the following:

(a) $\sin^{-1}(\frac{\sqrt{3}}{2})$ (b) $\sin^{-1}\left(\frac{-1}{2}\right)$ (c) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ $\cos^{-1}(\frac{-\sqrt{3}}{2})$ (d) $\tan^{-1}(\sqrt{3})$

(e) $\tan^{-1}(-\sqrt{3})$ (f) $\cot^{-1}(-1)$ (g) $sec^{-1}(-\sqrt{2})$ (h) $cosec^{-1}(-2)$

Solution: (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is an angle in $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ whose sine is $\frac{\sqrt{3}}{2}$. Thus $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$

(b) $\sin^{-1}\left(\frac{-1}{2}\right)$ is an angle in $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ whose sine is $-\frac{1}{2}$. Thus $\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

(c) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is an angle in $[0,\pi]$ whose cosine is $\frac{1}{\sqrt{2}}$. Thus $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$

(d) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is an angle in $[0,\pi]$ whose cosine is $-\frac{\sqrt{3}}{2}$.

Thus $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

(e) $\tan^{-1}(-\sqrt{3})$ is an angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is $-\sqrt{3}$. Thus $\tan^{-1}(-\sqrt{3})=-\frac{\pi}{3}$

(f) $\cot^{-1}(-1)$ is an angle in $(0,\pi)$ whose cotangent is -1 . Thus $\cot^{-1}(-1)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

(g) $sec^{-1}(-\sqrt{2})$ is an angle in $\left[0,\frac{\pi}{2}\right)\cup(0,\pi]$ whose secant is $-\sqrt{2}$. Thus

$sec^{-1}(-\sqrt{2})=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

(h) $cosec^{-1}(-2)$ is an angle in $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ whose co -secant is -2 . Thus

$cosec^{-1}(-2)=-\frac{\pi}{6}$

Example 2: Find the principal value of $\sin^{-1}(\sin 3)$

Solution: $\sin^{-1}(\sin 3)=\sin^{-1}\sin(\pi-3)=\pi-3$

Example 3: Find the principal value of $\sin^{-1}\sin(7)$

Solution: $\sin^{-1}\sin(7)=\sin^{-1}\sin(-2\pi+7)=-2\pi+7$

Example 4: Find the value of $\cos^{-1}\cos(5)$

Solution: $\cos^{-1}\cos(5)=\cos^{-1}\cos(2\pi-5)=2\pi-5$

Example 5: Find the principal value of $\cos^{-1}\cos 10$

Solution: $\cos^{-1}\cos 10=\cos^{-1}\cos[4\pi-10]=4\pi-10$

Example 6: Find the principal value of $\tan^{-1}\tan 8$

Solution: $\tan^{-1}\tan 8=\tan^{-1}(-3\pi+8)=-3\pi+8$

Example 7: Find the principal value of $\cos\left\{\sin^{-1}\frac{3}{5}\right\}$

Solution: Let $\sin^{-1}\frac{3}{5}=\theta\Rightarrow\sin\theta=\frac{3}{5}\Rightarrow\cos\theta=\frac{4}{5}\Rightarrow\cos\left\{\sin^{-1}\frac{3}{5}\right\}=\frac{4}{5}$

Example 8: Find $\cos\left\{\sin^{-1}-\frac{4}{5}\right\}$

Solution: $\sin^{-1}-\frac{4}{5}=\theta\Rightarrow\sin\theta=-\frac{4}{5}\Rightarrow\cos\theta=\frac{3}{5}\Rightarrow\cos\left\{\sin^{-1}-\frac{4}{5}\right\}=\frac{3}{5}$

Example 9: Find $\tan\left(\cos^{-1}\frac{5}{13}\right)$

Solution: Let $\cos^{-1}\frac{5}{13}=\theta \Rightarrow \cos\theta=\frac{5}{13}\Rightarrow \tan\theta=\frac{12}{5} \quad \Rightarrow \tan\left(\cos^{-1}\frac{5}{13}\right)=\frac{12}{5}$

Example 10: Find $\tan\left(\cos^{-1}-\frac{4}{5}\right)$

Solution: Let $\cos^{-1}-\frac{4}{5}=\theta \Rightarrow \cos\theta=-\frac{4}{5}\Rightarrow \tan\theta=-\frac{3}{4}\Rightarrow \tan\left(\cos^{-1}-\frac{4}{5}\right)=-\frac{3}{4}$

Example 11: Find $\sin\left(\tan^{-1}-\frac{3}{4}\right)$

Solution: Let $\tan^{-1}\left(-\frac{3}{4}\right)=\theta \quad \Rightarrow \tan\theta=-\frac{3}{4} \quad \Rightarrow \sin\theta=-\frac{3}{5}\Rightarrow \sin\left(\tan^{-1}-\frac{3}{4}\right)=-\frac{3}{5}$

Example 12: Find the value of $\sin^{-1}\left(\sin 4\frac{\pi}{3}\right)$

Solution: $\sin^{-1}\left(\sin 4\frac{\pi}{3}\right)=\sin^{-1}\left(\sin\left(\pi+\frac{\pi}{3}\right)\right)=\sin^{-1}\left(-\sin\frac{\pi}{3}\right)=-\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{-\pi}{3}$

Example 13: Find the value of

$\cos^{-1}\left(\cos 4\frac{\pi}{3}\right)=\cos^{-1}\left(\cos\left(\pi+\frac{\pi}{3}\right)\right)=\cos^{-1}\left(-\cos\frac{\pi}{3}\right)=\pi-\cos^{-1}\cos\frac{\pi}{3}=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

Example 14: Find the value of $\cos\left[\pi-\cos^{-1}\frac{1}{2}\right]$

Solution: $\cos\left[\pi-\cos^{-1}\frac{1}{2}\right]=\cos\left[\pi-\frac{\pi}{3}\right]=-\cos\frac{\pi}{3}=-\frac{1}{2}$

Example 15: Find the value of $\sin\left[2\sin^{-1}\frac{1}{\sqrt{2}}-\pi\right]$

Solution: $\sin\left[2\sin^{-1}\frac{1}{\sqrt{2}}-\pi\right]=\sin\left[2\times\frac{\pi}{4}-\pi\right]=\sin\left[\frac{\pi}{2}-\pi\right]=\sin\left[-\frac{\pi}{2}\right]=-1$

Example 16: Find the value of $\sin^{-1}(\cos 2)+\cos^{-1}(\cos 2)$.

Solution: As $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ therefore $\sin^{-1}(\cos 2)+\cos^{-1}(\cos 2)=\frac{\pi}{2}$

Example 17: Find the value of $\tan^{-1}\left[\frac{\pi}{2}+2\cot^{-1}(-1)\right]$

Solution: $\tan^{-1}\left[\frac{\pi}{2}+2\cot^{-1}(-1)\right]=\tan^{-1}\left[\frac{\pi}{2}+2\left(\pi-\cot^{-1}(-1)\right)\right) \quad =\tan^{-1}\left[\frac{\pi}{2}+2\left(\pi-\frac{\pi}{4}\right)\right]$

$=\tan^{-1}\left[\frac{\pi}{2}+2\times 3\frac{\pi}{4}\right]=\tan^{-1}2\pi=0$

Example 18: Prove that $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$

Solution: $\tan^{-1}1+\tan^{-2}+\tan^{-3}=\frac{\pi}{4}+\pi-\tan^{-1}\left\{\frac{2+3}{1-2\times 3}\right\}=\frac{\pi}{4}+\pi+\tan^{-1}(-1)=\frac{\pi}{4}+\pi-\frac{\pi}{4}=\pi$

Example19: Prove that $\tan^{-1}\frac{1}{3}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{13}{9}$

Solution: $\tan^{-1}\frac{1}{3}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{3}{4}=\tan^{-1}\left\{\frac{\frac{1}{3}+\frac{3}{4}}{1-\frac{1}{3}\times\frac{3}{4}}\right\}=\tan^{-1}\left\{\frac{\frac{13}{12}}{\frac{9}{12}}\right\}=\tan^{-1}\left(\frac{13}{9}\right)$

Example 20: Find $\tan(\cot^{-1}(-1))$

Solution: $\tan(\cot^{-1}(-1))=\tan(\pi-\cot^{-1}1)=\tan\left(\pi-\frac{\pi}{4}\right)=-\tan\frac{\pi}{4}=-1$

Example 21: Prove that $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{13}{9}$

Solution: As $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$ therefore $2\tan^{-1}\frac{1}{2}=\tan^{-1}\left[\frac{2\times\frac{1}{2}}{1-\frac{1}{2}^2}\right]=\tan^{-1}\frac{3}{4}$

Thus $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{13}{9}=\tan^{-1}\left\{\frac{\frac{1}{3}+\frac{3}{4}}{1-\frac{1}{3}\times\frac{3}{4}}\right\}=\tan^{-1}\left\{\frac{\frac{13}{12}}{\frac{9}{12}}\right\}=\tan^{-1}\left(\frac{13}{9}\right)$

Example 22 : Find $\tan^{-1}3-\tan^{-1}2$

Solution: $\tan^{-1}3-\tan^{-1}2=\tan^{-1}\frac{3-2}{1+2\times 3}=\tan^{-1}\frac{1}{6}$

Example 23: If a,b,c are positive real numbers such that $a<b<c$ then show that

$\cot^{-1}\frac{ab+1}{a-b}+\cot^{-1}\frac{bc+1}{b-c}+\cot^{-1}\frac{ca+1}{c-a}=2\pi$

Solution: As $a,b,c>0$ and $a>b>c$ therefore $\frac{ab+1}{a-b}<0,\frac{bc+1}{b-c}<0$ and $\frac{ca+1}{c-a}>0$

Thus $\cot^{-1}\frac{ab+1}{a-b}+\cot^{-1}\frac{bc+1}{b-c}+\cot^{-1}\frac{ca+1}{c-a}=\pi=\pi+\tan^{-1}\frac{a-b}{1+ab}+\pi+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}\frac{c-a}{1+ca}$

$=\pi+\tan^{-1}a-\tan^{-1}b+\pi+\tan^{-1}b-atn^{-1}c+atn^{-1}c+\tan^{-1}a=2\pi$

Example 24: If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$ then find the value of

$$x^{100}+y^{100}+z^{100}-\frac{9}{x^{200}+y^{200}+z^{200}}$$

Solution: As maximum value of $\sin^{-1}x=\frac{\pi}{2}$. Thus

$$\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}\Rightarrow\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}$$

$$\Rightarrow\sin^{-1}x=\sin^{-1}y=\sin^{¿}1\,z=\frac{\pi}{2}\Rightarrow x=y=z=\sin\frac{\pi}{2}=1$$

Thus $x^{100}+y^{100}+z^{100}-\frac{9}{x^{200}+y^{200}+z^{200}}=1+1+1-\frac{9}{1+1+1}=3-3=0$

Example 25: If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$ then find the value of $x^3+y^3+z^3-3xyz$

Solution: Maximum value of $\cos^{-1}x$ is π thus

$\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi\Rightarrow\cos^{-1}x=\cos^{-1}y=\cos^{-1}z=\pi \quad \Rightarrow x=y=z=\cos\pi=-1$

$\Rightarrow x^3+y^3+z^3-3xyz=-1-1-1-3(-1)(-1)(-1)=0$

Example 26: Prove that $2\sin^{-1}x=\sin^{-1}\{2x\sqrt{1-x^2}\};-\frac{1}{\sqrt{2}}\leq x\leq\frac{1}{\sqrt{2}}$

Solution: Put $\sin^{-1}x=\theta\Rightarrow x=\sin\theta$

Thus $\sin^{-1}\{2x\sqrt{1-x^2}\}=\sin^{-1}\{2\sin\theta\sqrt{1-\sin^2\theta}\}=\sin^{-1}\{2\sin\theta\sqrt{\cos^2\theta}\}$

$=\sin^{-1}(2\sin\theta\cos\theta)=\sin^{-1}\sin 2\theta=2\theta=2\sin^{-1}x$

Example 26: Prove that $\sin^{-1}(3x-4x^3)=3\sin^{-1}x;-\frac{1}{2}\leq x\leq\frac{1}{2}$

Solution: Put $\sin^{-1}x=\theta\Rightarrow\sin\theta=x$

Thus $\sin^{-1}(3x-4x^3)=\sin^{-1}(3\sin\theta-4\sin^3\theta)=\sin^{-1}\sin 3\theta=3\theta=3\sin^{-1}x$

Example: Prove that $2\cos^{-1}x=\cos^{-1}(2x^2-1);0<x\leq 1$

Solution: Let $\cos^{-1}x=\theta\Rightarrow x=\cos\theta$

Thus $\cos^{-1}(2x^2-1)=\cos^{-1}(2\cos^2\theta-1)=\cos^{-1}\cos 2\theta=2\theta=2\cos^{-1}x$

Example 27: Prove that $3\cos^{-1}x=\cos^{-1}\{4x^3-3x\};$ if $\frac{1}{2}\leq x\leq 1$

Solution: Let $\cos^{-1}x=\theta\Rightarrow x=\cos\theta\Rightarrow\cos^{-1}\{4x^3-3x\}=\cos^{-1}\{4\cos^3\theta-3\cos\theta\}=\cos^{-1}\cos 3\theta$

$=3\theta=3\cos^{-1}x$.

Example 28: Prove that $2\tan^{-1}x=\tan^{-1}\left(2\frac{x}{1-x^2}\right)$ if $-1<x<1$

Solution: Let $\tan^{-1}x=\theta\Rightarrow x=\tan\theta\Rightarrow\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\tan 2\theta=2\theta=2\tan^{-1}x$

Example 29: Prove that $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ if $-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

Solution: Let

$$\tan^{-1}x=\theta\Rightarrow x=\tan\theta\Rightarrow\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)=\tan^{-1}\tan 3\theta=3\theta=3\tan^{-1}x$$

Example 30: Prove that $2\tan^{-1}x=\sin^{-1}\left(2\frac{x}{1-x^2}\right);-1<x\le 1$

Solution: Let $\tan^{-1}x=\theta\Rightarrow x=\tan\theta\Rightarrow\sin^{-1}\left(\frac{2x}{1-x^2}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)=\sin^{-1}\sin 2\theta=2\theta=2\tan^{-1}x$

Example 31: Prove that $2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ if $o\le x<\infty$

Solution: Let $\tan^{-1}x=\theta\Rightarrow x=\tan\theta\Rightarrow\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}\cos 2\theta=2\theta=2\tan^{-1}x$

Example 32: Prove that $\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}=\frac{\pi}{4}+\frac{x}{2},0<x<\frac{\pi}{2}$

Solution: $\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}=\tan^{-1}\left\{\frac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}\right\}$

$$=\tan^{-1}\left\{\frac{\sqrt{2}\cos\frac{x}{2}+\sqrt{2}\sin\frac{x}{2}}{\sqrt{2}\cos\frac{x}{2}-\sqrt{2}\sin\frac{x}{2}}\right\}=\tan^{-1}\left\{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right\}=\tan^{-1}\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\}=\tan^{-1}\tan\left\{\frac{\pi}{4}+\frac{x}{2}\right\}=\frac{\pi}{4}+\frac{x}{2}$$

Example 33: Prove that $\tan^{-1}\left\{\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right\}=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$ if $-1<x<1$

Solution: Let $x^2=\cos\theta\Rightarrow\tan^{-1}\left\{\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right\}=\tan^{-1}\left\{\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right\}$

$$=\tan^{-1}\left\{\frac{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}}-\sqrt{2\sin^2\frac{\theta}{2}}}\right\}=\tan^{-1}\left\{\frac{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}\right\}=\tan^{-1}\left\{\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right\}=\tan^{-1}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$=\frac{\pi}{4}+\frac{\theta}{2}=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

Example 34: Prove that $\cos[\tan^{-1}(\sin(\cos^{-1}x))]=\sqrt{\frac{1+x^2}{1-x^2}}$

Solution: $\sin(\cot^{-1}x)=\sin\left\{\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right\}=\frac{1}{\sqrt{1+x^2}}$

$$\cos[\tan^{-1}(\sin(\cot^{-1}x))]=\cos\left\{\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right\}=\cos\left\{\cos^{-1}\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\}=\sqrt{\frac{1+x^2}{2+x^2}}$$

Example 35: Prove that $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}$

Solution: $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=2\left(\tan^{-1}\frac{1}{5}\right)-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$

$$=2\tan^{-1}\left\{\frac{2\times\frac{1}{5}}{1-\frac{1}{5}^2}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=2\tan^{-1}\frac{5}{12}-\left\{\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}\right\}$$

$$=\tan^{-1}\left\{\frac{2\times\frac{5}{12}}{1-\frac{5}{12}^2}\right\}-\tan^{-1}\left\{\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right\}=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{29}{6931}=\tan^{-1}\left\{\frac{\frac{20}{119}-\frac{29}{6931}}{1+\frac{120}{119}\times\frac{29}{6931}}\right\}=\tan^{-1}1=\frac{\pi}{4}$$

Example 36: If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$ then show that $\frac{x^2}{a^2}-\frac{2x\,y}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$

Solution: $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha\Rightarrow\cos^{-1}\left\{\frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right\}=\alpha \quad \Rightarrow\frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}=\cos\alpha$

$$\Rightarrow\frac{xy}{ab}-\cos\alpha=\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} \qquad \Rightarrow\left(\frac{xy}{ab}-\cos\alpha\right)^2=\left(\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right)^2$$

$$\Rightarrow\frac{x^2}{a^2}\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\alpha+\cos^2\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2}{a^2}\frac{y^2}{b^2} \qquad \Rightarrow\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\alpha=1-\cos^2\alpha$$

$$\Rightarrow\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\alpha=\sin^2\alpha$$

Example 37: Evaluate $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\}$

Solution: Let $\cos^{-1}\frac{\sqrt{3}}{5}=\theta\Rightarrow\cos\theta\frac{\sqrt{5}}{3}$

Now $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\}=\tan^{-1}\frac{\theta}{2}=\tan^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}=\sqrt{\frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}}}=\sqrt{\frac{3-\sqrt{5}}{3+\sqrt{5}}}$

$$=\sqrt{\frac{(3-\sqrt{5})^2}{(3+\sqrt{5})(3-\sqrt{5})}}=\sqrt{\frac{(3-\sqrt{5})^2}{(9-5)}}=\frac{3-\sqrt{5}}{2}$$

Example 38: Solve the equation $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

Solution: $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}\Rightarrow\tan^{-1}\left\{\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\times\frac{x+1}{x+2}}\right\}=\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left\{\frac{\frac{(x-1)(x+2)+(x-2)(x+1)}{(x-2)(x+2)}}{\frac{(x-2)(x+2)-(x-1)(x+1)}{(x-2)(x+2)}}\right\}=\frac{\pi}{4}\Rightarrow \tan^{-1}\left\{\frac{x^2+2x-x-2+x^2+x-2x-2}{x^2-4-(x^2-1)}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left\{\frac{2x^2-4}{-3}\right\}=\frac{\pi}{4}\Rightarrow\frac{2x^2-4}{-3}=\tan\frac{\pi}{4}\Rightarrow\frac{2x^2-4}{-3}=1\Rightarrow 2x^2-4=-3\Rightarrow 2x^2=1\Rightarrow x^2=\frac{1}{2}\Rightarrow x=\pm\sqrt{\frac{1}{2}}$$

Example 39: Solve the equation $2\tan^{-1}(\cos x)=\tan^{-1}(2\,cosec\,x)$

Solution: $2\tan^{-1}(\cos x)=\tan^{-1}(2\,cosec\,x)\Rightarrow \tan^{-1}\left\{\frac{2\cos x}{1-\cos^2 x}\right\}=\tan^{-1}(2\,cosec\,x)$

$$\Rightarrow\frac{2\cos x}{\sin^2}x=2\,cosec\,x\Rightarrow\cos x=\sin x\Rightarrow\tan x=1\Rightarrow x=\frac{\pi}{4}$$

Exercise

1. Find the principal values of the following:

   (a) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (b) $\cos^{-1}\frac{1}{2}$ $(c)\tan^{-1}(-\sqrt{3})$ (d) $\sin^{-1}\frac{1}{2}$ (e) $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

2. Find the principal value of

   (a) $\sin^{-1}\sin(3)$ (b) $\sin^{-1}\sin(11)$ (c) $\cos^{-1}\cos(7)$ $(d)\tan^{-1}(5)$

3. Find the value of the following:

   (a) $\sin^{-1}\sin\left(\frac{4\pi}{3}\right)$ (b) $\cos^{-1}\left(\frac{7\pi}{3}\right)$ (c) $\tan^{-1}\tan\left(2\pi-\frac{\pi}{4}\right)$

4. Find the value of $\sin\left(2\sin^{-1}\frac{3}{5}\right)$

5. Find the value of $\tan\left(3\tan^{-1}\frac{3}{4}\right)$

6. Find the value of $\cos\left(\pi+\cos^{-1}\left(-\frac{1}{2}\right)\right)$

7. Find the value of $\tan^{-1}1+\cot^{-1}\frac{1}{2}+\sin^{-1}\left(-\frac{1}{2}\right)$

8. Simplify $\tan^{-1}\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right);-\frac{\pi}{4}<x<\frac{\pi}{4}$

9. Simplify $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}};-\pi<x\pi$

10. Simplify $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right\};-\frac{\pi}{2}<x<\frac{\pi}{2}$

11. Simplify $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right);-\frac{\pi}{2}<x<\frac{\pi}{2}$

12. Simplify:

(a) $\sin^{-1}\left\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right\}$

(b) $\tan^{-1}\left\{x+\sqrt{1+x^2}\right\};x\in R$

(c) $\tan^{-1}\left\{\sqrt{1+x^2}-x\right\};x\in R$

(d) $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\};x\neq 0$

(e) $\tan^{-1}\left\{\frac{\sqrt{1+x^2}+1}{x}\right\};x\neq 0$

(f) $\tan^{-1}\sqrt{\frac{1-x}{1+x}};-1<x<1$

(g) $\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\};-a<x<a$

(h) $\tan^{-1}\left\{\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\};-\frac{\pi}{4}<x<\frac{\pi}{4}$

(i) $\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\};0<x<1$

(j) $\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$

13. Simplify $\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$

14. If $\cos^{-1}x+\sin^{-1}=\frac{3\pi}{2}$ then find the value of x^2+y^2+xy

15. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)-\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ then show that $x=\frac{a-b}{1+ab}$

16. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)=2\tan^{-1}x$ then show that $x=\frac{a+b}{1-ab}$

17. Prove that $\tan\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)=\frac{17}{6}$

18. Prove that $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$

19. Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}$

20. Prove that $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

21. Prove that $\cot^{-1}7+\cot^{-1}8+\cot_1 18=\cot^{-1}3$

22. Solve the following equations for x:

(a) $\tan^{-1}2\text{x}+\tan^{-1}3\text{x}=\frac{3\pi}{4}$

(b) $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$

(c) $3\sin^{-1}\frac{2\text{x}}{1+x^2}-4\cos^{-1}\frac{1-x^2}{1+x^2}+2\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$

23. Prove that $\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-sinx}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}=\frac{x}{2};0<x<\frac{\pi}{2}$

24. Prove that $\tan^{-1}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\}=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x \quad ;0<x<1$

25. Show that $\sin[\cot^{-1}[\cos[\tan^{-1}x]]]=\sqrt{\frac{x^2+1}{x^2+2}}$

26. Simplify $\tan^{-1}\left(\frac{a+bx}{b-ax}\right), x<\frac{b}{a}$

27. $\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right);-\frac{\pi}{2}<x<\frac{\pi}{2}, \frac{a}{b}\tan x>-1$

28. $\tan^{-1}\left(\frac{3\text{a}^x-x^3}{a^3-3ax^2}\right);-\frac{1}{\sqrt{3}}<x<\frac{x}{a}<\frac{1}{\sqrt{3}}$

29. Prove that $2\tan^{-1}\frac{1}{5}+sec^{-1}\frac{5\sqrt{2}}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

30. If $a>b>c>0$, prove that $\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)=\pi$