Matrices

Definition: A set of mn elements arranged in the form of a rectangular array of *m* rows and *n* columns is called an $m \times n$ matrix (to be read as *m* by *n* matrix).

An $m \times n$ matrix is usually written as $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{12} & a_{i3} & \cdots & a_{ij} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$

This can also be written as $A = [a_{ij}]_{m \times n}$

The numbers $a_{11}, a_{12}, a_{13}, \cdots$ are known as elements of the matrix. An element of i^{th} row and j^{th} column is represented by a_{ij}

Order of Matrix: If a matrix *A* has *m* rows and *n* columns then its order is defined as $m \times n$ and written as $O(A) = m \times n$

Number of elements: If a matrix has *m* rows and *n* columns then $n(A) = mn$

For example $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 4 & -6 \end{bmatrix}$ is a matrix of order 2×3 Here we have

$a_{11} = 2, a_{12} = 3, a_{13} = 4, a_{21} = 5, a_{22} = 4, a_{23} = -6$

Types of Matrices:

Row Matrix: It is a matrix having only one row.

For example $[1 \quad 2 \quad 3]$ is a row matrix of order 1×4

Column Matrix: It is a matrix having only one column.

For example $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column matrix of order 3×1

Square matrix: It is a matrix in which the number of rows is same as number of columns. If a square matrix has n rows and n columns then we say it is a square matrix of order n instead of saying a square matrix of order $n \times n$.

For example $\begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$ is a square matrix of order 2 and $\begin{bmatrix} 2 & 3 & -1 \\ 4 & 2 & 1 \\ -1 & 0 & 9 \end{bmatrix}$ is a square matrix of order 3

Diagonal element of a matrix: An element a_{ij} of a matrix is said to be diagonal element if and only if $i = j$. Thus a_{11}, a_{22}, a_{33} are diagonal elements.

Diagonal Matrix: A square matrix $A = [a_{ij}]_{n \times n}$ is said to be diagonal matrix if if all its diagonal elements are equal to zero. i.e. $a_{ij} = 0$ if $i = j$ If $d_1, d_2, d_3, \cdots d_n$ are diagonal elements of a diagonal matrix A of order n then we write $A = \text{diag}[d_1, d_2, d_3, \cdots, d_n]$

For example $A=\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & -1\end{bmatrix}$ is a diagonal matrix of order 3. A can also be written as $A=\text{diag}[1,2,-1]$

Scalar Matrix: A square matrix $A=[a_{ij}]_{n\times n}$ is said to be scalar matrix if all its diagonal elements are equal to non-zero constant and all non-diagonal elements are zero

or A square matrix $A=[a_{ij}]_{n\times n}$ is said to be scalar matrix if

(i) $a_{ij}=0 \text{ if } i\neq j$ and (ii) $a_{ij}=C$ whenever $i=j$ where $C\neq 0$

For example $\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 2 & 0\end{bmatrix}$ is a scalar matrix of order 3. It can also be written as $\text{diag}[2,2,2]$

Identity or Unit Matrix: A square matrix $A=[a_{ij}]_{n\times n}$ is said to be identity matrix if all its diagonal elements are equal to 1 and non-diagonal elements are zero.

i.e. $a_{ij}=1 \text{ if } i=j \text{ and } a_{ij}=0 \text{ if } i\neq j$

The identity matrix is generally denoted by I. In particular the identity matrix of order 2 is denoted by I_2 and that of order 3 is denoted by I_3.

Thus $I_2=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and $I_3=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

Null Matrix: It is a matrix whose all elements are equal to zero.

For example $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ are null matrices.

Lower triangular matrix A square matrix $A=[a_{ij}]_{n\times n}$ is said to be lower triangular if $a_{ij}=0 \text{ for all } i<j$

For example $\begin{bmatrix}1 & 0 & 0\\2 & 3 & 0\\1 & 2 & 3\end{bmatrix}$, $\begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}$ are lower triangular matrices of order 3 and 2 respectively

Upper triangular matrix A square matrix $A=[a_{ij}]_{n\times n}$ is said to be upper triangular if $a_{ij}=0 \text{ for all } i>j$

For example $\begin{bmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 6\end{bmatrix}$, $\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}$ are upper triangular matrices of order 3 and 2 respectively.

Equality of Matrices: Two matrices $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{r\times s}$ are said to be equal if and only if

(i) $m=r \text{ and } n=s$ and

(ii) $a_{ij}=b_{ij} \text{ for } i=1,2,3,\cdots,m \text{ and } j=1,2,3,\cdots,n$

That is two matrices are said to be equal if and only if they have same order and their corresponding elements are equal.

For example if $\begin{bmatrix} a & 2 \\ b & 4 \end{bmatrix} = \begin{bmatrix} 3 & x \\ 4 & y \end{bmatrix}$ then $a=3, b=4, x=2$ and $y=4$

Example1. A matrix has 15 elements. What are its possible order?
Solution: Possible orders are $1\times15, 3\times5, 5\times3, 15\times1$
Example2. A matrix has 11 elements. What are its possible order?
Solution: Possible orders are 1×11 or 11×1 .

Example 3: Construct a 3×3 matrix $A=[a_{ij}]$ whose elements are given by $a_{ij}=\frac{i-j}{i+j}$

Solution: $a_{11}=\frac{1-1}{1+1}=0, a_{12}=\frac{1-2}{1+2}=-\frac{1}{3}, a_{13}=\frac{1-3}{1+3}=-\frac{2}{4}=-\frac{1}{2}$

$a_{21}=\frac{2-1}{2+1}=\frac{1}{3}, a_{22}=\frac{2-2}{2+2}=0, a_{23}=\frac{2-3}{2+3}=-\frac{1}{5}$

$a_{31}=\frac{3-1}{3+1}=\frac{2}{4}=\frac{1}{2}, a_{32}=\frac{3-2}{3+2}=\frac{1}{5}, a_{33}=\frac{3-3}{3+3}=0$

Thus $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 & -\frac{1}{5} \\ \frac{1}{2} & \frac{1}{5} & 0 \end{bmatrix}$

Example 4: Construct a matrix $A=[a_{ij}]_{3\times3}$ where $a_{ij}=i+j$ if $i<j$ and $a_{ij}=i-j$ if $i\geqslant j$
Solution: $a_{11}=1-1=0, a_{12}=1+2=3, a_{13}=1+3=4$
$a_{21}=2-1=1, a_{22}=2-2=0, a_{23}=2+3=5, a_{31}=3-1=2, a_{32}=3-2=1, a_{33}=3-3=0$

Therefore $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & 5 \\ 2 & 1 & 0 \end{bmatrix}$

Example 5: Find the value of x, y, a and b if $\begin{bmatrix} 2x-y & a+b \\ 3a-2b & x+y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$

Solution: Two matrices are equal if they are of same order and their corresponding elements are equal

Therefore $\begin{bmatrix} 2x-y & a+b \\ 3a-2b & 2x+y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$

$\Rightarrow 2x-y=3, a+b=2, 3a-2b=1, 2x+y=5$
Solving $2x-y=3$ and $2x+y=5$ we get $x=2$ and $y=1$
Solving $a+b=2$ and $3a-2b=1$ we get $a=1$ and $b=1$

Exercise 3.1

1. If a matrix has 12 elements, what are its possible orders it can have? What if it has 5 elements?
2. If a matrix has p elements where p is prime number then show that it is either a row matrix or column matrix.
3. If $A=\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ then write the value of $a_{12}+a_{21}$.
4. If $A=\begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 4 & 5 & -7 \end{bmatrix}$ then write the sum of its diagonal elements
5. Construct a matrix of order 3×3 whose elements a_{ij} are given by
   (i) $a_{ij}=3i-j$ (ii) $a_{ij}=i^2+j$ (iii) $a_{ij}=\begin{Bmatrix} i^2+j \text{ if } i\leqslant j \\ i+j^2 \text{ if } i>j \end{Bmatrix}$
   (iv) $a_{ij}=\frac{|2i-3j|}{2}$ (v) $a_{ij}=\frac{2i+j^2}{2}$ $a_{ij}=i$
6. Find x, y, u, v if $\begin{bmatrix} 2x-3y & 2u+v \\ u-2v & 3x+y \end{bmatrix}=\begin{bmatrix} 7 & 3 \\ 4 & 5 \end{bmatrix}$
7. Find the value of x and y if $\begin{bmatrix} x+10 & y^2+2y \\ 0 & -4 \end{bmatrix}=\begin{bmatrix} 3x+4 & 3 \\ 0 & y^2-5y \end{bmatrix}$
8. Give an example of a row matrix which also a column matrix.
9. Give an example of diagonal matrix.
10. Is every diagonal matrix a scalar matrix also? Is every scalar matrix a diagonal matrix also?
11. Give an example of lower triangular matrix. Also give an example of upper triangular matrix.

Addition of Matrices: Let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{r\times s}$ then $A+B$ exists if and only if $m=r$ and' n =s i.e. $O(A)=O(B)$ and if $C=A+B$ then $c_{ij}=a_{ij}+b_{ij}$ for all $i=1,2,3,\ldots m$ and $j=1,2,3,\ldots n$

For example $if\ A=\begin{bmatrix} 1 & 3 & -2 \\ 2 & 3 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 5 & 0 \\ -3 & 4 & 5 \end{bmatrix}$ then $A+B=\begin{bmatrix} 3 & 8 & -2 \\ -1 & 7 & 4 \end{bmatrix}$

Properties of Matrix Addition:

Commutative: If A and B are two matrices of same order then $A+B=B+A$

Associative: If A, B and C are three matrices of same order then $A+(B+C)=(A+B)+C$

Existence of Identity: The null matrix is the identity element for the matrix addition.
$i.e.\ A+O=O+A=O$

Existence of Inverse: For every matrix $A=[a_{ij}]_{m\times n}$ there exists one and only one matrix $[-a_{ij}]_{m\times n}$ denoted by $-A$ such that $A+(-A)=O=(-A)+A$
$-A$ is known as additive inverse of A

Cancellation Laws: If A, B and C are the matrices of same order then $A+B=A+C \Rightarrow B=C$

Multiplication of a Matrix by a scalar: If $A=[a_{ij}]_{m\times n}$ and k be any scalar then kA is a matrix obtained by multiplying each element of A by k

Thus $kA=[ka_{ij}]_{m\times n}$

For example if $A=\begin{bmatrix}2 & 3 & 4\\ -1 & 2 & 3\end{bmatrix}$ then $2A=\begin{bmatrix}4 & 6 & 8\\ -2 & 4 & 6\end{bmatrix}$

If $A=\begin{bmatrix}2 & 4\\ 5 & 3\end{bmatrix}$ then $-\frac{1}{2}A=\begin{bmatrix}-1 & -2\\ -\frac{5}{2} & -\frac{3}{2}\end{bmatrix}$

Subtraction of two matrices: If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ then $A-B=[a_{ij}-b_{ij}]_{m\times n}$

For example if $A=\begin{bmatrix}3 & 4 & 5\\ 5 & -4 & 5\\ 1 & 0 & 9\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1 & 1\\ 7 & 6 & 2\\ -1 & 2 & -3\end{bmatrix}$ then $A-B=\begin{bmatrix}1 & 3 & 4\\ -2 & -10 & 3\\ 2 & -2 & 12\end{bmatrix}$

Example1 : Find a matrix X such that $A+B+X=0$ where $A=\begin{bmatrix}2 & 1 & -3\\ 3 & -2 & 4\\ 5 & 2 & 3\end{bmatrix}$ and

$B=\begin{bmatrix}1 & -7 & 2\\ 3 & 5 & 1\\ -2 & 0 & 1\end{bmatrix}$

Solution: As $A+B+X=0\Rightarrow X=-(A+B)=-\begin{bmatrix}3 & -6 & -1\\ 6 & 3 & 5\\ 3 & 2 & 4\end{bmatrix}=\begin{bmatrix}3 & 6 & 1\\ -6 & -3 & -5\\ -3 & -2 & -4\end{bmatrix}$

Example2: If $A=\begin{bmatrix}2 & 3\\ 5 & 2\end{bmatrix}$ and $B=\begin{bmatrix}3 & 2\\ 5 & -2\end{bmatrix}$ then find $2A-3B$

Solution: $2A=\begin{bmatrix}4 & 6\\ 10 & 4\end{bmatrix}$ and $3B=\begin{bmatrix}9 & 6\\ 15 & -6\end{bmatrix}$ $\Rightarrow 2A-3B=\begin{bmatrix}-5 & 0\\ -5 & 10\end{bmatrix}$

Example 3: Find matrices A and B if $A+B=\begin{bmatrix}2 & 4\\ 6 & 2\end{bmatrix}$ and $A-B=\begin{bmatrix}4 & 6\\ 0 & 8\end{bmatrix}$

Solution: We have $A+B=\begin{bmatrix}2 & 4\\ 6 & 2\end{bmatrix}$ and $A-B=\begin{bmatrix}4 & 6\\ 0 & 8\end{bmatrix}$

Adding these two equations we get $A+B+A-B=\begin{bmatrix}2 & 4\\ 6 & 2\end{bmatrix}+\begin{bmatrix}4 & 6\\ 0 & 8\end{bmatrix}$

$\Rightarrow 2A=\begin{bmatrix}6 & 10\\ 6 & 10\end{bmatrix}\Rightarrow A=\frac{1}{2}\begin{bmatrix}6 & 10\\ 6 & 10\end{bmatrix}=\begin{bmatrix}3 & 5\\ 3 & 5\end{bmatrix}$

Now $A+B-(A-B)=\begin{bmatrix}2 & 4\\ 6 & 2\end{bmatrix}-\begin{bmatrix}4 & 6\\ 0 & 8\end{bmatrix}$

$\Rightarrow 2B=\begin{bmatrix}-2 & -2\\ 6 & -6\end{bmatrix}\Rightarrow B=\frac{1}{2}\begin{bmatrix}-2 & -2\\ 6 & -6\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 3 & -3\end{bmatrix}$

Example 4: Find a matrix A such that $A+2\begin{bmatrix}2 & -1\\3 & 2\end{bmatrix}+\begin{bmatrix}1 & 3\\4 & -5\end{bmatrix}=\begin{bmatrix}0 & 1\\5 & 4\end{bmatrix}$

Solution: $A+2\begin{bmatrix}2 & -1\\3 & 2\end{bmatrix}+\begin{bmatrix}1 & 3\\4 & -5\end{bmatrix}=\begin{bmatrix}0 & 1\\5 & 4\end{bmatrix}\Rightarrow A+\begin{bmatrix}4 & -2\\6 & 4\end{bmatrix}+\begin{bmatrix}1 & 3\\4 & -5\end{bmatrix}=\begin{bmatrix}0 & 1\\5 & 4\end{bmatrix}$

$\Rightarrow A+\begin{bmatrix}5 & 1\\10 & -1\end{bmatrix}=\begin{bmatrix}0 & 1\\5 & 4\end{bmatrix}\Rightarrow A=\begin{bmatrix}0 & 1\\5 & 4\end{bmatrix}-\begin{bmatrix}5 & 1\\10 & -1\end{bmatrix}\Rightarrow A=\begin{bmatrix}-5 & 0\\-5 & 5\end{bmatrix}$

Exercise 3.2

1. Find a matrix X such that $A+B+X=0$ where $A=\begin{bmatrix}1 & -2 & 3\\5 & -2 & 4\\7 & 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 6 & 4\\0 & 12 & 3\\-1 & 4 & 5\end{bmatrix}$
2. Find matrices A and B such that $2A+3B=\begin{bmatrix}2 & -1 & 4\\0 & 2 & 3\\-7 & 2 & 1\end{bmatrix}$ and $A+2B=\begin{bmatrix}1 & 0 & 5\\4 & 1 & 6\\2 & 3 & 1\end{bmatrix}$
3. Find x, y, z, t if $2\begin{bmatrix}x & z\\y & t\end{bmatrix}+3\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}=3\begin{bmatrix}3 & 5\\4 & 6\end{bmatrix}$
4. If $A=\begin{bmatrix}12 & 3\\5 & 0\end{bmatrix}, B=\begin{bmatrix}2 & -1\\-6 & 4\end{bmatrix}$ then find the value of $2A+3B$
5. Find matrices A and B if $A+B=\begin{bmatrix}2 & 3 & 4\\-4 & 5 & 1\end{bmatrix}$ and $A-B=\begin{bmatrix}3 & -1 & 12\\4 & 0 & -9\end{bmatrix}$
6. If $A=\begin{bmatrix}1 & -3 & 2\\2 & 0 & 2\end{bmatrix}, B=\begin{bmatrix}2 & -1 & -1\\1 & 0 & 1\end{bmatrix}$, find the matrix C such that $A+B+C=0$
7. Find the value of λ such that $\lambda\begin{bmatrix}2 & 1\\3 & 1\\-1 & 4\end{bmatrix}+\begin{bmatrix}0 & 1\\4 & 5\\2 & 8\end{bmatrix}=\begin{bmatrix}-1 & 2\\0 & 7\\4 & 6\end{bmatrix}$
8. Find the value of x and y satisfying $\begin{bmatrix}x-y & 2 & -2\\4 & x & 6\end{bmatrix}+\begin{bmatrix}3 & -2 & 2\\1 & 0 & -1\end{bmatrix}=\begin{bmatrix}6 & 0 & 0\\5 & 2x+y & 5\end{bmatrix}$

Multiplication of Matrices

Definition: If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ then AB exists and its order is $m\times p$ and

$(AB)_{ij}=\sum_{r=1}^{n} a_{ir}b_{rj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$

Example 1: Let $A=\begin{bmatrix}2 & 3 & 4\\2 & 1 & 4\end{bmatrix}$ $B=\begin{bmatrix}1 & 3\\-4 & 5\\2 & 3\end{bmatrix}$

Solution: Since A is 2×3 matrix and B is 3×2 matrix therefore AB is a 2×2 matrix.

Let $AB=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ Then we have

$c_{11}=2\times1+3\times-4+4\times2=2-12+8=-2$
$c_{12}=2\times3+3\times5+4\times3=6+15+12=33$
$c_{21}=2\times1+1\times-4+4\times2=2-4+8=6$
$c_{22}=2\times3+1\times5+4\times3=6+5+12=23$

Thus $AB=\begin{bmatrix} -2 & 33 \\ 6 & 23 \end{bmatrix}$

Example 2: Let $A=\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ then find AB .

Solution: As order of A is 3×1 and that of B is 1×3 thus AB exists and its order is 1×1
Thus $AB=[1\times4+2\times5+3\times6]=[4+10+18]=[32]$

Example 3: Let $A=\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 \\ 3 & -1 \\ -4 & 5 \end{bmatrix}$. Find AB

Solution: As $O(A)=1\times3$ and $O(B)=3\times2$ thus AB exists and its order is 1×2 .
$AB=\begin{bmatrix} 1\times2+2\times3+3\times-4 & 1\times1+2\times-1+3\times5 \end{bmatrix}=\begin{bmatrix} -4 & 14 \end{bmatrix}$

Example 14: Let $A=\begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ -3 & 2 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 \\ 3 & -2 \\ 4 & 5 \end{bmatrix}$. Find AB

Solution: As A is a matrix of order 3×3 and B is a matrix of order 3×2 .
Therefore AB is a matrix of order 3×2

$$AB=\begin{bmatrix} 1\times2+2\times3+(-1)\times4 & 1\times1+2\times-2+(-1)\times5 \\ 2\times2+(-3)\times3+4\times4 & 2\times1+(-3)\times-2+4\times5 \\ (-3)\times2+2\times3+0\times4 & (-3)\times1+2\times-2+0\times5 \end{bmatrix}=\begin{bmatrix} 4 & -8 \\ 11 & 28 \\ 0 & -7 \end{bmatrix}$$

Example 5: Find AB if $A=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$

Solution: A is a matrix of order 3×1 and B is a matrix of order 1×3 thus AB is a matirx of order 3×3

Thus $AB=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 4 & 5 & 6 \end{bmatrix}=\begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$

Example 6: If $A=\begin{bmatrix}1 & -2 & 3\\ 2 & 0 & 1\\ 3 & -1 & -2\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1 & 4\\ 2 & 5 & 1\\ 0 & 1 & 4\end{bmatrix}$ then find AB

Solution: $AB=\begin{bmatrix}1 & -2 & 3\\ 2 & 0 & 1\\ 3 & -1 & -2\end{bmatrix}\begin{bmatrix}2 & -1 & 4\\ 2 & 5 & 1\\ 0 & 1 & 4\end{bmatrix}=\begin{bmatrix}-2 & -8 & 14\\ 4 & -1 & 12\\ 4 & -10 & 3\end{bmatrix}$

Example 7: If $A=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}$ then show that $A^2-7\text{A}-2I=0$ where I is the identity matrix of order 2

Solution: $A^2=A\times A=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}=\begin{bmatrix}16 & 21\\ 28 & 37\end{bmatrix}$

$7A=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}=\begin{bmatrix}14 & 21\\ 28 & 35\end{bmatrix}$ and $2I=2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$

Therefore $A^2-7A-2I=\begin{bmatrix}16 & 21\\ 28 & 37\end{bmatrix}-\begin{bmatrix}14 & 21\\ 28 & 35\end{bmatrix}-\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}-\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$

Example 8: If $A=\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}$ then find the value of λ and μ such that $A^2+\lambda A+\mu I=0$

Solution: $A^2=\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}=\begin{bmatrix}7 & 18\\ 6 & 19\end{bmatrix}$

$\lambda A=\begin{bmatrix}2\lambda & 3\lambda\\ \lambda & 4\lambda\end{bmatrix}$ and $\mu I=\begin{bmatrix}\mu & 0\\ 0 & \mu\end{bmatrix}$

Thus $A^2+\lambda A+\mu I=0\Rightarrow\begin{bmatrix}7 & 18\\ 6 & 19\end{bmatrix}+\begin{bmatrix}2\lambda & 3\lambda\\ \lambda & 4\lambda\end{bmatrix}+\begin{bmatrix}\mu & 0\\ 0 & \mu\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$\begin{bmatrix}7+2\lambda+\mu & 18+3\lambda\\ 6+\lambda & 19+4\lambda+\mu\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ $\Rightarrow 2\lambda+\mu+7=0;\ 18+3\lambda=0;\ 6+\lambda=0$ and $4\lambda+\mu+19=0$

$\Rightarrow\lambda=-6$ and $\mu=5$

Example 9: If $A=\begin{bmatrix}1 & 3\\ -2 & 4\end{bmatrix}$ and $f(x)=x^2+6x+2$

Solution: Therefore $f(x)=x^2+6x+2$

$\Rightarrow f(A)=A^2+6A+2I=\begin{bmatrix}1 & 3\\ -2 & 4\end{bmatrix}\begin{bmatrix}1 & 3\\ -2 & 4\end{bmatrix}+6\begin{bmatrix}1 & 3\\ -2 & 4\end{bmatrix}+2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow f(A)==\begin{bmatrix}-5 & 15\\ -10 & 10\end{bmatrix}+\begin{bmatrix}6 & 18\\ -12 & 24\end{bmatrix}+\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix}3 & 33\\ -22 & 36\end{bmatrix}$

Example 10: If $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ then show that $A^2=\begin{bmatrix}\cos 2\theta & -\sin 2\theta\\ \cos 2\theta & \sin 2\theta\end{bmatrix}$

Solution: $A^2=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}\cos^2\theta-\sin^2\theta & -2\sin\theta\cos\theta\\ 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}=\begin{bmatrix}\cos 2\theta & -2\sin\theta\\ \sin 2\theta & \cos 2\theta\end{bmatrix}$

Example 11: Let $A=\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}$ and I be identity matrix of order 2. Show that

$$I+A=(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

Solution: We have $I+A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}=\begin{bmatrix}1 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 1\end{bmatrix}$

and $I-A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}=\begin{bmatrix}1 & \tan\frac{\alpha}{2}\\ -\tan\frac{\alpha}{2} & 1\end{bmatrix}$

Therefore $(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & \tan\frac{\alpha}{2}\\ -\tan\frac{\alpha}{2} & 1\end{bmatrix}=\begin{bmatrix}\dfrac{1-\tan^2\frac{\alpha}{2}}{+\tan^2\frac{\alpha}{2}} & -\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\\ \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \dfrac{1-\tan^2\frac{\alpha}{2}}{+\tan^2\frac{\alpha}{2}}\end{bmatrix}$

$$\Rightarrow(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & t\\ -t & 1\end{bmatrix}\begin{bmatrix}\dfrac{1-t^2}{1+t^2} & \dfrac{-2t}{1+t^2}\\ \dfrac{2t}{1+t^2} & \dfrac{1-t^2}{1+t^2}\end{bmatrix}=\begin{bmatrix}\dfrac{1-t^2+2t^2}{1+t^2} & \dfrac{-2t+t-t^3}{1+t^2}\\ \dfrac{-t+t^3+2t}{1+t^2} & \dfrac{2t^2+1-t^2}{1+t^2}\end{bmatrix}$$

$$\Rightarrow(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}\dfrac{1+t^2}{1+t^2} & \dfrac{-t(1+t^2)}{1+t^2}\\ \dfrac{t(1+t^2)}{1+t^2} & \dfrac{1+t^2}{1+t^2}\end{bmatrix}=\begin{bmatrix}1 & -t\\ t & 1\end{bmatrix}$$

$$\Rightarrow (I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 1\end{bmatrix}=I+A$$

Example 12: Find the value of x if $\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}=0$

Solution: $\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}=0$

$$\Rightarrow\begin{bmatrix}2x+16 & 5x+6 & x+4\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}=0\Rightarrow[2x+16+10x+12+x^2+4x]=0$$

$$\Rightarrow x^2+16x+28=0\Rightarrow(x+14)(x+2)=0\Rightarrow x=-2 \text{ or } x=-14$$

Example 13: Let $A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ then show that $A^n=\begin{bmatrix}\cos n\theta & \sin n\theta\\ -\sin n\theta & \cos n\theta\end{bmatrix}$ for all natural numbers n

Solution: We shall prove the result by principle of mathematical induction.

For $n=1$, we have $A^1=\begin{bmatrix}\cos 1\theta & \sin 1\theta\\ -\sin 1\theta & \cos 1\theta\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

Thus result is true for $n=1$

Let result is true for $n=k$ or let $A^k=\begin{bmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{bmatrix}$

Now $A^{k+1}=A^k\times A=\begin{bmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{bmatrix}\times\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

$$\Rightarrow A^{k+1}=\begin{bmatrix}\cos k\theta\cos\theta-\sin k\theta\sin\theta & \cos k\theta\sin\theta+\sin k\theta\cos\theta\\ -(\sin k\theta\cos\theta+\cos k\theta\sin\theta) & -\sin k\theta\sin\theta\sin\theta+\cos k\theta\cos\theta\end{bmatrix}$$

$$\Rightarrow A^{k+1}=\begin{bmatrix}\cos(k+1)\theta & \sin(k+1)\theta\\ -\sin(k+1)\theta & \cos(k+1)\theta\end{bmatrix}$$

Thus result is true for $n=k+1$ also.

Hence by Principle of Mathematical Induction result is true for every natural number n

Hence $A^n=\begin{bmatrix}\cos n\theta & \sin n\theta\\ -\sin n\theta & \cos n\theta\end{bmatrix}$ for every natural number n

Example 14: Prove the following by principle of mathematical induction:

If $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, then $A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$ for every natural number n

Solution: When $n=1$ we have $A^1=A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=\begin{bmatrix}1+2(1) & -4(1)\\ 1 & 1-2(1)\end{bmatrix}$

So the result is true for $n=1$ Let the result be true for $n=k$. Then,

$$A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}$$

Now we will show the result is true for $n=k+1$

$$A^{k+1}=\begin{bmatrix}1+2(k+1) & -4(k+1)\\ k+1 & 1-2(k+1)\end{bmatrix}$$

$$A^{k+1}=A^k\times A=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=\begin{bmatrix}3+6k-4k & -4-8k+4k\\ 3k+1-2k & -4k-1+2k\end{bmatrix}$$

$$\Rightarrow A^{k+1}=\begin{bmatrix}3+2k & -4-4k\\ k+1 & -1-2k\end{bmatrix}=\begin{bmatrix}1+2(k+1) & -4(k+1)\\ k+1 & 1-2(k+1)\end{bmatrix}$$

This shows that the result is true for $n=k+1$, whenever it is true for $n=k$.

Hence by principle of mathematical induction the result is true for any natural number n

Example 15: If $A=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$ then prove that $(aI+bA)^n=a^nI+na^{n-1}bA$ where I is unit matrix of order 2 and n is any positive number.

Solution: We shall prove the result by principle of mathematical induction.

When $n=1$ we have $(aI+bA)^1=aI+bA=a^1I+1a^0bA=a^1I+1a^{1-1}bA$

So, the result is true for $n=1$

Let the result be true for $n=k$. Then

$$(aI+bA)^k=a^kI+ka^{k-1}bA$$

Now we shall show that the result is true for $n=k+1$ .i.e.

$$(aI+bA)^{k+1}=a^{k+1}I+(k+1)a^kbA$$

Now $(aI+bA)^{k+1}=(aI+bA)^k(aI+bA)=(a^kI+ka^{k-1}bA)(aI+bA)$

$$(aI+bA)^{k+1}=(a^kI)(aI)+(a^kI)(bA)+(ka^{k-1}bA)(aI)+(ka^{k-1}bA)(bA)$$

$$(aI+bA)^{k+1}=(a^ka)(II)+a^kb(IA)+ka^kb(AI)+ka^{k-1}b^2(AA)$$

$$(aI+bA)^{k+1}=a^{k+1}I+a^kbA+ka^kbA+ka^{k-1}b^2A^2 \quad [\text{as } IA=AI \text{ and } II=I]$$

$$(aI+bA)^{k+1}=a^{k+1}I+(ka^kb+a^kb)A+ka^{k-1}b^2A^2$$

But $A^2=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$

$$\Rightarrow(aI+bA)^{k+1}=a^{k+1}I+(k+1)a^kbA$$

This shows that the result is true for $n=k+1$, whenever it is true for $n=k$.

Hence by principle of mathematical induction the result is true for any natural number n

Example 16: Prove that the product of matrices

$\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$ and $\begin{bmatrix}\cos^2\phi & \cos\phi\sin\phi\\ \cos\phi\sin\phi & \sin^2\phi\end{bmatrix}$ is the null matrix, when θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$

We have $\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}\begin{bmatrix}\cos^2\phi & \cos\phi\sin\phi\\ \cos\phi\sin\phi & \sin^2\phi\end{bmatrix}$

$=\begin{bmatrix}\cos^2\theta\cos^2\phi+\cos\theta\cos\phi\sin\theta\sin\phi & \cos^2\theta\cos\phi\sin\phi+\cos\theta\sin\theta\sin^2\phi\\ \cos^2\phi\cos\theta\sin\theta+\sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi+\sin^2\theta\sin^2\phi\end{bmatrix}$

$=\begin{bmatrix}\cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi)\\ \sin\theta\cos\phi\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi)\end{bmatrix}$

$=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ $\left(\text{as}\,\theta-\phi=(2n+1)\frac{\pi}{2}, n\in Z\Rightarrow\cos(\theta-\phi)=0\right)$

Example 17: If $A=\begin{bmatrix}\cos x+\sin x & \sqrt{2}\sin x\\ -\sqrt{2}\sin x & \cos x-\sin x\end{bmatrix}$, prove that

$A^n=\begin{bmatrix}\cos nx+\sin nx & \sqrt{2}\sin nx\\ -\sqrt{2}\sin nx & \cos nx-\sin nx\end{bmatrix}$ for all $n\in N$

Solution: Let us prove the result using the principle of mathematical induction.

For $n=1$ we have $A^1=\begin{bmatrix}\cos x+\sin x & \sqrt{2}\sin x\\ -\sqrt{2}\sin x & \cos x-\sin x\end{bmatrix}=\begin{bmatrix}\cos 1x+\sin 1x & \sqrt{2}\sin 1x\\ -\sqrt{2}\sin 1x & \cos 1x-\sin 1x\end{bmatrix}$

So, result is true for $n=1$.

Let result is true for $n=k$. Then

$A^k=\begin{bmatrix}\cos kx+\sin kx & \sqrt{2}\sin kx\\ -\sqrt{2}\sin kx & \cos kx-\sin kx\end{bmatrix}$

We shall prove that result is true for $n=k+1$ .ie.

$A^{k+1}=\begin{bmatrix}\cos(k+1)x+\sin(k+1)x & \sqrt{2}\sin(k+1)x\\ -\sqrt{2}\sin(k+1)x & \cos(k+1)x-\sin(k+1)x\end{bmatrix}$

Now $A^{k+1}=A^kA=\begin{bmatrix}\cos kx+\sin kx & \sqrt{2}\sin kx\\ -\sqrt{2}\sin kx & \cos kx-\sin kx\end{bmatrix}\begin{bmatrix}\cos x+\sin x & \sqrt{2}\sin x\\ -\sqrt{2}\sin x & \cos x-\sin x\end{bmatrix}$

$\Rightarrow A^{k+1}=\begin{bmatrix}(\cos kx+\sin kx)(\cos x+\sin x)-2\sin kx\sin x & \sqrt{2}\sin x(\cos kx+\sin kx)+\sqrt{2}\sin kx(\cos x-\sin x)\\ -\sqrt{2}\sin kx(\cos x+\sin x)-\sqrt{2}\sin x(\cos kx-\sin kx) & -2\sin kx\sin x+(\cos kx-\sin kx)(\cos x-\sin x)\end{bmatrix}$

$=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ (say)

Then $a=\cos kx\cos x+\cos kx\sin x+\sin kx\cos x+\sin kx\sin x-2\sin kx\sin x$

$=\cos kx\cos x-\sin kx\sin x+\cos kx\sin x+\sin kx\sin x=\cos(k+1)x+\sin(k+1)x$

$b=\sqrt{2}(\sin x\cos kx+\sin x\sin kx+\sin kx\cos x-\sin kx\sin x)=\sqrt{2}(\sin(k+1)x)$

$c=-\sqrt{2}(\sin kx\cos x+\sin kx\sin x+\sin x\cos kx-\sin x\sin kx)=-\sqrt{2}(\sin(k+1)x)$

$d=-2\sin kx\sin x+\cos kx\cos x-\cos kx\sin x-\sin kx\cos x+\sin kx\sin x$

$=\cos kx\cos x-\sin kx\sin x-(\sin kx\, sinx+\cos kx\sin x)=\cos(k+1)x-\sin(k+1)x$

$\Rightarrow\quad A^{k+1}=\begin{bmatrix}\cos(k+1)x+\sin(k+1)x & \sqrt{2}\sin(k+1)x\\ -\sqrt{2}\sin(k+1)x & \cos(k+1)x-\sin(k+1)x\end{bmatrix}$

Thus we have proved the result is true for $n=k+1$ whenever it is true for $n=k$.

Hence by principle of mathematical induction result is true for all $n \in N$

Exercise 3.3

1. If $A=\begin{bmatrix}1 & 2\\4 & 2\end{bmatrix}, B=\begin{bmatrix}-2 & 0\\4 & 5\end{bmatrix}$ then find AB and BA. Is $AB=BA$.
2. Give an example of two matrices A and B such that $AB=0$
3. Let $A=\begin{bmatrix}3 & 2\\-1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1\\5 & 2\end{bmatrix}$ then compute $(A+B)^2$ and $A^2+2AB+B^2$. Are these equal?
4. Find the matrix A such that $A\cdot\begin{bmatrix}2 & -1\\1 & 0\\-3 & 4\end{bmatrix}=\begin{bmatrix}-1 & -8 & -10\\1 & -2 & -5\\9 & 22 & 15\end{bmatrix}$
5. Solve for x and y given that $\begin{bmatrix}x & y\\3y & x\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\5\end{bmatrix}$
6. If $A=\begin{bmatrix}2 & -2 & -4\\-1 & 3 & 4\\1 & -2 & -3\end{bmatrix}$, show that $A^2=A$
7. If $A=\begin{bmatrix}4 & -1 & -4\\3 & 0 & -4\\3 & -1 & -3\end{bmatrix}$, show that $A^2=I$
8. If $A=\begin{bmatrix}2 & -3 & -5\\-1 & 4 & 5\\1 & -3 & -4\end{bmatrix}$ and $B=\begin{bmatrix}2 & -2 & -4\\-1 & 3 & 4\\1 & -2 & -3\end{bmatrix}$, show that $AB=A$ and $BA=B$
9. If $A=\begin{bmatrix}ab & b^2\\-a^2 & -ab\end{bmatrix}$, show that $A^2=O$
10. Find the value of x if $\begin{bmatrix}x & 4 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 2\\1 & 0 & 2\\0 & 2 & -4\end{bmatrix}\begin{bmatrix}x\\4\\-1\end{bmatrix}=0$, find x
11. Find x if $\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\2 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=0$, find x

12. If $A=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, show that $A^2+6A+11I=0$

13. If $A=\begin{bmatrix} 5 & 3 \\ -1 & 2 \end{bmatrix}$, show that $A^2-7A+13I=0$

14. If $A=\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$ then find the value of k such that $A^2+kA+2I=0$

15. If $A=\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ then find the value of k such that $A^2-kA+4I=0$

16. If $A=\begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}$ then find the value of λ and μ such that $A^2+\lambda A+\mu I=0$

17. If $A=\begin{bmatrix} 3 & -1 \\ 2 & -7 \end{bmatrix}$ then find λ and μ such that $A^2+\lambda A+\mu I=0$

18. If $A=\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, $f(x)=x^2-3x+5$, show that $f(A)=0$

19. If $f(x)=\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $f(x)=x^2-2x-3$, show that $f(A)=0$

20. If $A=\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$, find A^{32}

21. If $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, prove that $A^n=\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for all positive integers n.

22. If $A=\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, prove that $A^n=\begin{bmatrix} a^n & \dfrac{b(a^n-1)}{a-1} \\ 0 & 1 \end{bmatrix}$ for all positive integers n.

23. If $A=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, prove that $A^n=\begin{bmatrix} 1 & n & \dfrac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$ for all positive integers n.

24. If $A=\begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$, show that $A^n=\begin{bmatrix} \cos n\theta & i\sin n\theta \\ i\sin n\theta & \cos n\theta \end{bmatrix}$ for all positive integers n

25. If $A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, show that $A^n=\begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$ for all positive integers n

26. If $A_\alpha=\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that

(i) $A_\alpha \cdot A_\beta = A_{\alpha+\beta}$

(ii) $(A_\alpha)^n = \begin{bmatrix} \cos n\alpha & -\sin n\alpha & 0 \\ \sin n\alpha & \cos n\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $n \in N$

27. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, then show that A is zero of the polynomial $f(x) = x^3 - 6x^2 + 7x + 2$

Transpose of a Matrix:

Definition: Let $A = [a_{ij}]_{m \times n}$ matrix, then transpose of A is denoted by A^T and is given by $A^T = [a_{ij}]_{m \times n}$. Thus if A is a $m \times n$ matrix then A^T is a a $n \times m$ matrix obtained from A by interchanging rows into columns.

For example if $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ then $A^T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 3 \end{bmatrix}$.

If $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 1 & 7 \end{bmatrix}$ then $B^T = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 5 & 7 \end{bmatrix}$.

Results

(i) $(kA)^T = kA^T$

$(ii)\ (A+B)^T = A^T + B^T$

$(iii)\ (A-B)^T = A^T - B^T$

$(iv)\ (A^T)^T = A$

$(v)\ (AB)^T = B^T A^T$

$(vi)\ \ (ABC)^T = C^T B^T A^T$

Symmetric Matrix: A square matrix $A = [a_{ij}]_{n \times n}$ is said to be symmetric if $A^T = A$ i.e if $a_{ij} = a_{ji}\ for\ i = 1$ to n

For example $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 4 \\ 1 & 4 & 6 \end{bmatrix}$ is a symmetric matrix as $A^T = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 4 \\ 1 & 4 & 6 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 4 \\ 1 & 4 & 6 \end{bmatrix} = A$.

Skew-Symmetric Matrix: A square matrix $A = [a_{ij}]_{n \times n}$ is said to be skew-symmetric if $A^T = -A$ i.e $a_{ij} = -a_{ji}$ for $i = 1$ to n

For example $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ is a skew-symmetric matrix as $A^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} = -A$

Example1: If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix}$ then verify that $(AB)^T = B^T\ \ A^T$

Solution: $AB=\begin{bmatrix}2 & 3\\ -1 & 4\end{bmatrix}\begin{bmatrix}-2 & 1\\ 5 & 2\end{bmatrix}=\begin{bmatrix}11 & 8\\ 22 & 7\end{bmatrix}$

$\Rightarrow (AB)^T=\begin{bmatrix}11 & 22\\ 8 & 7\end{bmatrix}$

$B^T A^T=\begin{bmatrix}-2 & 5\\ 1 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}=\begin{bmatrix}11 & 22\\ 8 & 7\end{bmatrix}$

$\Rightarrow (AB)^T=B^T A^T$

Example 2: If $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ then show that $AA^T=I$.

Solution: $AA^T=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}\cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\sin\theta\cos\theta\\ \sin\theta\cos\theta-\cos\theta\sin\theta & \sin^2\theta+\cos^2\theta\end{bmatrix}$

$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

Example 3: Show that the elements on the main diagonal of a skew-symmetric matrix are all zero.

Solution: Let $A=[a_{ij}]$ be a skew-symmetric matrix. Then

$a_{ij}=a_{ji}$ for all i, j

$\Rightarrow a_{ii}=-a_{ii}$ for all values of i $\Rightarrow 2a_{ii}=0$ for all values of i

$\Rightarrow a_{ij}=0$ for all values of i

$a_{11}=a_{22}=a_{33}=\cdots=a_{nn}=0$

Example 4: For any square matrix A prove that

(i) $A+A^T$ is symmetric matrix

(ii) $A-A^T$ is skew-symmetric matrix.

Solution: (i) Let $P=A+A^T \quad \Rightarrow \quad P^T=(A+A^T)^T = A^T+(A^T)^T$ (As $(A+B)^T=A^T+B^T$)

$P^T=A^T+A$ {As $(A^T)^T=A$ }

$P^T=A+A^T=A$

Therefore P is a symmetric matrix.

Let $Q=A-A^T$. Then

$Q^T=(A-A^T)^T=A^T-(A^T)^T$ (As $(A-B)^T=A^T-B^T$)

$\Rightarrow \quad Q^T=A^T-A$ (As $(A^T)^T=A$)

$\Rightarrow Q^T=-(A-A)^T=-Q$

$\therefore$ Q is a skew-symmetric matrix.

Example 4: Prove that every square matrix A can be uniquely expressed as a sum of symmetric matrix and skew-symmetric matrix.

Solution: Let $P=\frac{1}{2}(A+A^T)$ and $Q=\frac{1}{2}(A-A^T)$.

Then $P+Q=\frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)=A$

Now $P^T = \left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A + A^T)^T$ (As $(kA)^T = kA^T$)

$\Rightarrow\ P^T = \frac{1}{2}(A^T + (A^T)^T)$ (As $(A+B)^T = A^T + B^T$)

$\Rightarrow P^T = \frac{1}{2}(A^T + A)$ (As $(A^T)^T = A$)

$\Rightarrow\ P^T = \frac{1}{2}(A + A^T) = P$

$\therefore$ P is symmetric matrix.

$Q^T = \left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - (A^T)^T)$

$\Rightarrow\ Q^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -Q$

$\Rightarrow\ Q$ is skew-symmetric matrix.

Thus $A = P + Q$ where, P is symmetric matrix and Q is skew-symmetric matrix.

Thus A has been expressed as a sum of symmetric and skew-symmetric matrix.

Now let if possible let $A = R + S$ where R is symmetric matrix and Q is skew-symmetric matrix.

Therefore $A^T = (R+S)^T = R^T + S^T$

$\Rightarrow\ A^T = R - S$ ($\because$ $R^T = R$ and $S^T = -S$)

Now $A = R + S$ and $A^T = R - S$

$\Rightarrow\ R = \frac{1}{2}(A + A^T) = P$ and $S = \frac{1}{2}(A - A^T) = Q$

Hence every square matrix A can %beta uniquely expressed as a sum of symmetric matrix and skew-symmetric matrix.

Note: For any square matrix A

(i) $\frac{1}{2}(A + A^T)$ is known as symmetric part of A

(ii) $\frac{1}{2}(A - A^T)$ is known as skew-symmetric part of A

Example 5: Find the symmetric and skew-symmetric part of $\begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 5 \\ 0 & -1 & 4 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 5 \\ 0 & -1 & 4 \end{bmatrix}$

$\Rightarrow\ A^T = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & -1 \\ -4 & 5 & 4 \end{bmatrix}$

Now $A+A^T=\begin{bmatrix} 4 & 4 & -4 \\ 4 & 4 & 4 \\ -4 & 4 & 8 \end{bmatrix}$ and $A-A^T=\begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix}$

Therefore symmetric part of $A=\frac{1}{2}(A+A^T)=\begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & 2 \\ -2 & 2 & 4 \end{bmatrix}$

and skew-symmetric part of $A=\frac{1}{2}(A-A^T)=\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

Example 6: Express $A=\begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & -1 \\ 2 & 6 & 7 \end{bmatrix}$ as a sum of symmetric and skew-symmetric matrices.

Solution: $A=\begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & -1 \\ 2 & 6 & 7 \end{bmatrix} \Rightarrow A^T=\begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 6 \\ 1 & -1 & 7 \end{bmatrix}$

$\therefore \quad A+A^T=\begin{bmatrix} 6 & 6 & 3 \\ 6 & 10 & 5 \\ 3 & 5 & 14 \end{bmatrix}$ and $A-A^T=\begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -7 \\ 1 & 7 & 0 \end{bmatrix}$

$\therefore$ Symmetric part of $A=\frac{1}{2}(A-A^T)=\begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 2 & \frac{5}{2} & -\frac{1}{2} \\ 1 & 3 & \frac{7}{2} \end{bmatrix}$

Skew-Symmetric part of $A=\frac{1}{2}(A-A^T)=\begin{bmatrix} 0 & \frac{1}{2} & -1 \\ -\frac{1}{2} & 0 & \frac{3}{2} \\ 1 & -\frac{3}{2} & 0 \end{bmatrix}$

Thus $A=\begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 2 & \frac{5}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{7}{2} \end{bmatrix}+\begin{bmatrix} 0 & \frac{1}{2} & -1 \\ -\frac{1}{2} & 0 & \frac{3}{2} \\ 1 & -\frac{3}{2} & 0 \end{bmatrix}$

Example 7: Prove that ABA^T is symmetric according as B is symmetric or skew-symmetric.

Solution: Let $P=ABA^T$

$\Rightarrow \quad P^T=(ABA^T)^T=(A^T)^T B^T A^T \qquad (\because ((ABC)^T=C^T B^T A^T)$

$\Rightarrow \quad P^T=AB^T A^T \qquad (\because (A^T)^T=A) \qquad \cdots(i)$

Let B be symmetric matrix. Then $B^T=B \qquad \cdots(ii)$

Thus by (i) and (ii)we get $P^T=ABA^T=P$

$\therefore$ P is symmetric matrix.

Let B be skew-symmetric matrix. Then $B^T=-B \qquad \cdots$ (iii)

Thus by (I) and (iii) we get $P^T=A(-B^T)A^T=-ABA^T=-P$

$\therefore$ P is skew-symmetric matrix.

Thus ABA^T is symmetric and skew-symmetric according as B is symmetric and skew-symmetric matrix.

Example 8: Prove that skew-symmetric part of a symmetric matrix is zero.

Solution: Let A be any symmetric matrix. Then $A^T=A$

$\therefore$ Skew-Symmetric Part of $A=\frac{1}{2}(A-A^T)=\frac{1}{2}(A-A)=0$

Example 9: Prove that symmetric part of a skew-symmetric matrix is zero.

Solution: Let A be any skew-symmetric matrix. Then $A^T=-A$

$\therefore$ Symmetric Part of $A=\frac{1}{2}(A+A^T)=\frac{1}{2}(A-A)=0$

Example 10: Prove that AA^T is symmetric matrix.

Solution: Let $P=AA^T$. Then

$P^T=(AA)^T=(A^T)^T A^T \qquad (\because (AB)^T=B^T A^T)$

$\Rightarrow \quad P^T=AA^T \qquad$ (As $(A^T)^T=A$)

Thus P is symmetric matrix.

Exercise: 3.4

1. Prove that $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{bmatrix}$ is a skew-symmetric matrix.
2. Write the sum of diagonal elements of a skew-symmetric matrix.
3. If A is a skew-symmetric matrix of order 3 then find the value of $a_{23}+a_{32}$.

4. If $A=\begin{bmatrix} 2 & 3 & 1 \\ -3 & 2 & 4 \\ 0 & 1 & 7 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 1 & 4 \\ 0 & -1 & 5 \\ 2 & 6 & 1 \end{bmatrix}$ then verify that $(AB)^T=B^T A^T$

5. If $A=\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ then verify that $(AB)^T=B^T A^T$

6. If $A=\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then verify that $AA^T=I$

7. Express $A=\begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 7 & -1 & 1 \end{bmatrix}$ as a sum of symmetric and skew-symmetric matrix.

8. Find the value of k such that $A=\begin{bmatrix} 2 & 3 & k \\ 3 & 4 & 5 \\ 2k-1 & 5 & 6 \end{bmatrix}$ is a symmetric matrix.

9. Find the symmetric and skew-symmetric part of $A=\begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 4 \\ 1 & 5 & 6 \end{bmatrix}$.

10. If A is a symmetric matrix of order 3 and $a_{23}=2$ then find the value of a_{32}

Inverse of a Matrix

Definition: A square matrix A of order n is said to be invertible if there exists a square matrix of order B of order n such that $AB=BA=I$ where I is identity matrix of order n. We write $A^{-1}=B$

Thus $AA^{-1}=A^{-1}A=I$

Elementary row Transformation:

(i) $R_i \Leftrightarrow R_j$

(ii) $R_i \rightarrow R_i+\alpha R_j$

Elementary Column Transformation:

(i) $C_i \Leftrightarrow C_j$

(ii) $C_i \rightarrow C_i+\alpha C_j$

Elementary Matrix: A matrix obtained by an identity matrix by applying a single elementary operation (row transformation or column transformation) is known as elementary matrix.

Result: Let $C=AB$ be a product of two matrices. Any elementary row(column) transformation of AB can be obtained by subjecting the pre-factor A(post-factor B) to the same elementary row(column) transformation.

Example1. Find the inverse of the matrix $A=\begin{bmatrix}2 & 1 & 3\\ 1 & 3 & 2\\ 3 & 4 & 1\end{bmatrix}$ by elementary row transformation.

Solution: Let $A=\begin{bmatrix}2 & 1 & 3\\ 1 & 3 & 2\\ 3 & 4 & 1\end{bmatrix}$

Write $A=IA$ i.e. $\begin{bmatrix}2 & 1 & 3\\ 1 & 3 & 2\\ 3 & 4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$

Applying $R_1\to\frac{1}{2}R_1$, $R_3\to\frac{1}{3}R_3$ we get

$$\begin{bmatrix}1 & \frac{1}{2} & \frac{3}{2}\\ 1 & 3 & 2\\ 1 & \frac{4}{3} & \frac{1}{3}\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{3}\end{bmatrix}A$$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$ we get

$$\begin{bmatrix}1 & \frac{1}{2} & \frac{3}{2}\\ 0 & \frac{5}{2} & \frac{1}{2}\\ 0 & \frac{5}{6} & -\frac{7}{6}\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 0 & 0\\ -\frac{1}{2} & 1 & 0\\ -\frac{1}{2} & 0 & \frac{1}{3}\end{bmatrix}A$$

Applying $R_1\to 2R_1$, $R_2\to\frac{2}{5}R_2$ and $R_3\to\frac{6}{5}R_3$ we get

$$\begin{bmatrix}2 & 1 & 3\\ 0 & 1 & \frac{1}{5}\\ 0 & 1 & -\frac{7}{5}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -\frac{1}{5} & \frac{2}{5} & 0\\ -\frac{3}{5} & 0 & \frac{2}{5}\end{bmatrix}A$$

Applying $R_1\to R_1-R_2$ and $R_3\to R_3-R_2$ we get

$$\begin{bmatrix}2 & 0 & \frac{14}{5}\\ 0 & 1 & \frac{1}{5}\\ 0 & 0 & -\frac{8}{5}\end{bmatrix}=\begin{bmatrix}\frac{6}{5} & -\frac{2}{5} & 0\\ -\frac{1}{5} & \frac{2}{5} & 0\\ -\frac{2}{5} & -\frac{2}{5} & \frac{2}{5}\end{bmatrix}A$$

$R_1 \to \frac{5}{14} R_1, R_2 \to 5R_2$ and $R_3 \to -\frac{5}{8} R_3$ we get

$$\begin{bmatrix} \frac{5}{7} & 0 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & -\frac{1}{7} & 0 \\ -1 & 2 & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$ we get

$$\begin{bmatrix} \frac{5}{7} & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{28} & -\frac{11}{28} & \frac{1}{4} \\ -\frac{5}{4} & \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$$

Applying $R_1 \to \frac{7}{5} R_1$, $R_2 \to \frac{1}{5} R_2$ we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{11}{20} & \frac{7}{20} \\ -\frac{1}{4} & \frac{7}{20} & \frac{1}{20} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$$

$$\Rightarrow I = \begin{bmatrix} \frac{1}{4} & -\frac{11}{20} & \frac{7}{20} \\ -\frac{1}{4} & \frac{7}{20} & \frac{1}{20} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$$

Thus $A^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{11}{20} & \frac{7}{20} \\ -\frac{1}{4} & \frac{7}{20} & \frac{1}{20} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

Example-2: Find the inverse of the matrix $\begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & 4 & 2 \end{bmatrix}$ by elementary row transformation.

Solution: Let $A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & 4 & 2 \end{bmatrix}$ and write $A = IA$ i.e.

$$\begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to \frac{1}{2} R_{1,}\ R_2 \to \frac{1}{3} R_{2,}\ R_3 \to \frac{1}{3} R_3$ we get

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{3}{2} \\ 1 & \frac{2}{3} & -\frac{1}{3} \\ 1 & \frac{4}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} A$$

Applying we get $R_2 \to R_2 - R_{1,}\ R_3 \to R_3 - R_1$ we get

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & -\frac{5}{6} & -\frac{11}{6} \\ 0 & -\frac{1}{6} & -\frac{5}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{3} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{3} \end{bmatrix} A$$

$R_1 \to \frac{2}{3} R_{1,}\ R_2 \to -\frac{6}{5} R_{2,}\ R_3 \to -6 R_3$

$$\begin{bmatrix} \frac{2}{3} & 1 & 1 \\ 0 & 1 & \frac{11}{5} \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{3}{5} & -\frac{2}{5} & 0 \\ 3 & 0 & -2 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_{2,}\ R_3 \to R_3 - R_2$ we get

$$\begin{bmatrix} \frac{2}{3} & 0 & -\frac{6}{5} \\ 0 & 1 & \frac{11}{5} \\ 0 & 0 & \frac{14}{5} \end{bmatrix} = \begin{bmatrix} -\frac{4}{15} & \frac{2}{5} & 0 \\ \frac{3}{5} & -\frac{2}{5} & 0 \\ \frac{12}{5} & \frac{2}{5} & -2 \end{bmatrix} A$$

Applying $R_1 \to -\frac{5}{6} R_{1,} R_2 \to \frac{5}{11} R_{2,} R_3 \to \frac{5}{14} R_3$ we get

$$\begin{bmatrix} -\frac{5}{9} & 0 & 1 \\ 0 & \frac{5}{11} & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & -\frac{1}{3} & 0 \\ \frac{3}{11} & -\frac{2}{11} & 0 \\ \frac{6}{7} & \frac{1}{7} & -\frac{5}{7} \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_{3,} R_2 \to R_2 - R_3$ we get

$$\begin{bmatrix} -\frac{5}{9} & 0 & 0 \\ 0 & \frac{5}{11} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{40}{63} & -\frac{10}{21} & \frac{5}{7} \\ -\frac{45}{77} & -\frac{25}{77} & \frac{5}{7} \\ \frac{6}{7} & \frac{1}{7} & -\frac{5}{7} \end{bmatrix} A$$

Applying $R_1 \to -\frac{9}{5} R_{1,} R_2 \to \frac{11}{5} R_2$ we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{8}{7} & \frac{6}{7} & -\frac{9}{7} \\ -\frac{9}{7} & -\frac{5}{7} & \frac{11}{7} \\ \frac{6}{7} & \frac{1}{7} & -\frac{5}{7} \end{bmatrix} A$$

Example-3: Find the inverse of the matrix $\begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 2 \\ 4 & 1 & 2 \end{bmatrix}$ by elementary column transformation.

Solution: Let $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 2 \\ 4 & 1 & 2 \end{bmatrix}$ and write $A = AI$ or

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 2 \\ 4 & 1 & 2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_1 \to \frac{1}{2} C_1$, $C_3 \to \frac{1}{4} C_3$ we get

$$\begin{bmatrix} 1 & 1 & 1 \\ \frac{3}{2} & -1 & \frac{1}{2} \\ 2 & 1 & \frac{1}{2} \end{bmatrix} = A \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ we get

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & -\frac{5}{2} & -1 \\ 2 & -1 & -\frac{3}{2} \end{bmatrix} = A \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$C_1 \to \frac{3}{2} C_1$, $C_2 \to -\frac{2}{5} C_2$ and $C_3 \to -C_3$ we get

$$\begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 1 & 1 & 1 \\ \frac{4}{3} & \frac{2}{5} & \frac{3}{2} \end{bmatrix} = A \begin{bmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{2} \\ 0 & -\frac{2}{5} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}$$

$C_1 \to C - 1 - C_{2,} C_3 \to C_3 - C_2$ we get

$$\begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \\ \frac{14}{15} & \frac{2}{5} & \frac{11}{10} \end{bmatrix} = A \begin{bmatrix} \frac{2}{15} & \frac{1}{5} & \frac{3}{10} \\ \frac{2}{5} & -\frac{2}{5} & \frac{2}{5} \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}$$

Applying $C_1 \to \frac{15}{14} C_{1,} C_2 \to \frac{5}{2} C_{2,} C_3 \to \frac{10}{11} C_3$ we get

$$\begin{bmatrix} \frac{5}{7} & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} = A \begin{bmatrix} \frac{1}{7} & \frac{1}{2} & \frac{3}{11} \\ \frac{3}{7} & -1 & \frac{4}{11} \\ 0 & 0 & -\frac{5}{22} \end{bmatrix}$$

Applying $C_1 \to C_1 - C_{3,} C_2 \to C_2 - C - 3$ we get

$$\begin{bmatrix} \frac{5}{7} & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -\frac{10}{77} & \frac{5}{22} & \frac{3}{11} \\ \frac{5}{77} & -1511 & \frac{4}{11} \\ \frac{5}{22} & \frac{5}{22} & -\frac{5}{22} \end{bmatrix}$$

Applying $C_1 \to \frac{7}{5} C_{1,} C_2 \to \frac{2}{5} C_2$ we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -\frac{2}{11} & \frac{1}{11} & \frac{3}{11} \\ \frac{1}{11} & -\frac{6}{11} & \frac{4}{11} \\ \frac{7}{22} & \frac{1}{11} & -\frac{5}{22} \end{bmatrix}$$

Thus $I = A \begin{bmatrix} -\frac{2}{11} & \frac{1}{11} & \frac{3}{11} \\ \frac{1}{11} & -\frac{6}{11} & \frac{4}{11} \\ \frac{7}{22} & \frac{1}{11} & -\frac{5}{22} \end{bmatrix}$

Thus $A^{-1} = \begin{bmatrix} -\frac{2}{11} & \frac{1}{11} & \frac{3}{11} \\ \frac{1}{11} & -\frac{6}{11} & \frac{4}{11} \\ \frac{7}{22} & \frac{1}{11} & -\frac{5}{22} \end{bmatrix}$

Example-4 Find the inverse of the matrix $\begin{bmatrix} 2 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ by elemntary column transformation .

Solution: Let $A=\begin{bmatrix}2 & 2 & 3\\3 & 2 & 1\\4 & 2 & 1\end{bmatrix}$. Write $A=AI$ or

$$\begin{bmatrix}2 & 2 & 3\\3 & 2 & 1\\4 & 2 & 1\end{bmatrix}=A\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 0\end{bmatrix}$$

Applying $C_1\to\frac{1}{2}C_{1,}C_2\to\frac{1}{2}C_{2,}C_3\to\frac{1}{3}C_3$ we get

$$\begin{bmatrix}1 & 1 & 1\\\frac{3}{2} & 1 & \frac{1}{3}\\2 & 1 & \frac{1}{3}\end{bmatrix}=A\begin{bmatrix}\frac{1}{2} & 0 & 0\\0 & \frac{1}{2} & 0\\0 & 0 & \frac{1}{3}\end{bmatrix}$$

Applying $C_2\to C_2-C_{1,}C_3\to C_3-C_1$ we get

$$\begin{bmatrix}1 & 0 & 0\\\frac{3}{2} & -\frac{1}{2} & -\frac{7}{6}\\2 & -1 & -\frac{5}{3}\end{bmatrix}=A\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\\0 & \frac{1}{2} & 0\\0 & 0 & \frac{1}{3}\end{bmatrix}$$

Applying $C_1\to\frac{2}{3}C_{1,}C_2\to-2C_{1,}C_3\to-\frac{6}{7}C_3$ we get

$$\begin{bmatrix}\frac{2}{3} & 0 & 0\\1 & 1 & 1\\\frac{4}{3} & 2 & \frac{10}{7}\end{bmatrix}=A\begin{bmatrix}\frac{1}{3} & 1 & \frac{3}{7}\\0 & -1 & 0\\0 & 0 & -\frac{2}{7}\end{bmatrix}$$

Applying $C_1\to C_1-C_{2,}C_2\to C_3-C_2$ we get

$$\begin{bmatrix}\frac{2}{3} & 0 & 0\\0 & 1 & 0\\-\frac{2}{3} & 2 & -\frac{4}{7}\end{bmatrix}=A\begin{bmatrix}-\frac{2}{3} & 1 & -\frac{4}{7}\\1 & -1 & 1\\0 & 0 & -\frac{2}{7}\end{bmatrix}$$

Applying $C_1\to-\frac{3}{2}C_{1,}C_2\to\frac{1}{2}C_{2,}C_3\to-\frac{7}{4}C_3$ we get

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ -\frac{3}{2} & -\frac{1}{2} & -\frac{7}{4} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Applying $C_1 \to C_1 - C - 3, C_2 \to C_2 - C_3$ we get

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & -\frac{1}{2} & 1 \\ \frac{1}{4} & \frac{5}{4} & -\frac{7}{4} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Applying $C_1 \to -C_{1,} C_2 \to 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & -1 & 1 \\ -\frac{1}{4} & \frac{5}{2} & -\frac{7}{4} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

$$I = A \begin{bmatrix} 0 & -1 & 1 \\ -\frac{1}{4} & \frac{5}{2} & -\frac{7}{4} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -\frac{1}{4} & \frac{5}{2} & -\frac{7}{4} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$

Example-5: Find the inverse of the matrix $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

Write $A = IA$ or

$$\begin{bmatrix}0 & 1 & 3\\1 & 2 & 1\\2 & 1 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}A$$

Applying $R_1 \Leftrightarrow R_1$ we get

$$\begin{bmatrix}1 & 2 & 1\\0 & 1 & 3\\2 & 1 & 3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}A$$

Applying $R_2 \rightarrow R_3 - 2R_1$ we get

$$\begin{bmatrix}1 & 2 & 1\\0 & 1 & 3\\0 & -3 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & -2 & 1\end{bmatrix}A$$

Applying $R_1 \rightarrow R_1 - 2R_{2,} R_3 \rightarrow R_3 + 3R_2$ we get

$$\begin{bmatrix}1 & 0 & -5\\0 & 1 & 3\\0 & 0 & 10\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\1 & 0 & 0\\3 & -2 & 1\end{bmatrix}A$$

Applying $R_3 \rightarrow \frac{1}{10}R_3$ we get

$$\begin{bmatrix}1 & 0 & -5\\0 & 1 & 3\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\1 & 0 & 0\\\frac{3}{10} & -\frac{1}{5} & \frac{1}{10}\end{bmatrix}A$$

Applying $R_1 \rightarrow R_1 + 5R_{3,} R_2 \rightarrow R_2 - 3R_3$ we get

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{2} & 0 & \frac{1}{2}\\\frac{1}{10} & \frac{3}{5} & -\frac{3}{10}\\\frac{3}{10} & -\frac{1}{5} & \frac{1}{10}\end{bmatrix}A$$

$$I=\begin{bmatrix}-\frac{1}{2} & 0 & \frac{1}{2}\\\frac{1}{10} & \frac{3}{5} & -\frac{3}{10}\\\frac{3}{10} & -\frac{1}{5} & \frac{1}{10}\end{bmatrix}A$$

Hence $A^{-1}=\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{10} & \frac{3}{5} & -\frac{3}{10} \\ \frac{3}{10} & -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$

Example-6: If $A=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then show that $A^2-7A-2I=0$ and hence find A^{-1}.

Solution: $A^2=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}=\begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix}$, $7\text{A}=7\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}=\begin{bmatrix} 14 & 21 \\ 28 & 35 \end{bmatrix}$, $2\text{I}=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Thus $A^2-7\text{A}-2\text{I}=\begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix}-\begin{bmatrix} 14 & 21 \\ 28 & 35 \end{bmatrix}-\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus $A^2-7A-2I=0$

Pre-Multiply by A^{-1} we get

$A^{-1}A^2-7A^{-1}A-2A^{-1}A=0 \Rightarrow A-7I-2A^{-1}=0$

$\Rightarrow 2\text{A}^{-1}=A-7\text{I}=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}-\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}=\begin{bmatrix} -5 & 3 \\ 4 & -2 \end{bmatrix}$

$\Rightarrow A^{-1}=\frac{1}{2}\begin{bmatrix} -5 & 3 \\ 4 & -2 \end{bmatrix}$

$\Rightarrow A^{-1}=\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$

Example-7: If $A=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ then find AB and hence find A^{-1}

Solution: $AB=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}=\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Thus $AB=4I$

Pre-Multiply by $A^{-1}AB=4A^{-1}I \Rightarrow IB=4A^{-1} \Rightarrow B=4A^{-1} \Rightarrow A^{-1}=\frac{1}{4}B$

Thus $A^{-1}=\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$

Exercise 3.4

1. Find the inverse of the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 3 & -1 & 2 \end{bmatrix}$ by elementary ow transformation.
2. Find the inverse of the matrix $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ by elementary row transformation.
3. Find the inverse of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ -1 & 0 & 4 \end{bmatrix}$ by elementary row transformation.
4. Find the inverse of the $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix}$ by elementary row transformation.
5. Find the inverse of the matrix $\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$ by elementary column transformation.
6. Find the inverse of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ 2 & -1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$ by elementary column transformation.
7. Find the inverse of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ by elementary column transformation.
8. If $A=\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ then show that $A^2-7A+11I=0$ and hence find A^{-1}
9. If $A=\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} -5 & 10 & 1 \\ -3 & 6 & -1 \\ -10 & 12 & 2 \end{bmatrix}$ then find AB and find A^{-1}

Answer Ex:3.1

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(1) $1\times12, 2\times6, 2\times4, 4\times3, 6\times2, 12\times1$ $1\times5, 5\times1$ (3) 2 (4) -3

(5) (i) $\begin{bmatrix} 2 & 1 & 0 \\ 5 & 4 & 3 \\ 8 & 7 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 10 & 11 & 12 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 6 & 11 \\ 10 & 11 & 12 \end{bmatrix}$ (iv) $\begin{bmatrix} \frac{1}{2} & 2 & \frac{7}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \\ \frac{3}{2} & 0 & \frac{3}{2} \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 6 & 11 \\ 5 & 8 & 13 \\ 7 & 10 & 12 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Answer Ex 3.2

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(1) $\begin{bmatrix} 0 & -4 & -7 \\ -5 & -10 & -7 \\ -6 & -5 & -5 \end{bmatrix}$ (2) $A=\begin{bmatrix} 1 & -2 & -7 \\ -12 & 1 & -12 \\ -20 & -5 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 1 & 6 \\ 8 & 0 & 9 \\ 11 & 4 & 1 \end{bmatrix}$

(3) $x=3, y=6, z=9, t=6$ (4) $\begin{bmatrix} 30 & -3 \\ -8 & 12 \end{bmatrix}$ (5) A= $\begin{bmatrix} \frac{5}{2} & 1 & 8 \\ 0 & \frac{5}{2} & -4 \end{bmatrix}$ and B= $\begin{bmatrix} -\frac{1}{2} & 2 & -4 \\ -4 & \frac{5}{2} & 5 \end{bmatrix}$

(6) $C=\begin{bmatrix} -3 & 4 & -1 \\ -3 & 0 & -3 \end{bmatrix}$ (7) $x=\frac{3}{2}$, $y=-\frac{3}{2}$

Answer Ex 3.3

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(1) $AB=\begin{bmatrix} 6 & 10 \\ 0 & 10 \end{bmatrix}$ and $BA=\begin{bmatrix} -2 & -4 \\ 24 & 33 \end{bmatrix}$ (5) $x=1, y=1$ (11) $x=-2$ (14) $k=3$ (15) $k=5$

(16) $\lambda=-7$ and $\mu=12$ (17) $\lambda=4$, $\mu=-19$ (20) $A^2=O$

Answer Ex 3.4

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(7) $A=\begin{bmatrix} 3 & 3 & \frac{11}{2} \\ 3 & 2 & \frac{5}{2} \\ \frac{11}{2} & \frac{5}{2} & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -\frac{3}{2} \\ 2 & 0 & \frac{7}{2} \\ \frac{3}{2} & -\frac{7}{2} & 0 \end{bmatrix}$

(8) Symmetric part = $\begin{bmatrix} 3 & \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & 1 & \frac{9}{2} \\ \frac{5}{2} & \frac{9}{2} & 6 \end{bmatrix}$ and skew-symmetric part = $\begin{bmatrix} 0 & -\frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & 0 \end{bmatrix}$

Ex-3.4

(1) $\begin{bmatrix} \frac{5}{6} & -\frac{7}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{7}{6} & \frac{11}{6} & \frac{1}{6} \end{bmatrix}$ (2) $\begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{3}{7} & \frac{2}{7} \end{bmatrix}$ (3) $\begin{bmatrix} \frac{4}{25} & \frac{4}{25} & -\frac{1}{5} \\ -\frac{11}{25} & \frac{14}{25} & -\frac{1}{5} \\ \frac{1}{25} & \frac{1}{25} & \frac{1}{5} \end{bmatrix}$ (4) $\begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{7}{9} & \frac{2}{3} & -\frac{4}{9} \\ \frac{1}{9} & -\frac{1}{3} & \frac{2}{9} \end{bmatrix}$

(5) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{bmatrix}$ (6) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} \\ -1 & \frac{10}{7} & \frac{1}{7} \end{bmatrix}$ (7) $\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$ (8) $\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$

(9) $\begin{bmatrix} -\frac{5}{8} & \frac{5}{4} & \frac{1}{8} \\ -\frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix}$