# Relations and Functions

**Cartesian Product of two sets:** If A and B are two sets then their Cartesian product is denoted by $A\times B$ and is given by $A\times B=\{(a,b): a\in A \text{ and } b\in B\}$

Example: Consider two sets $A=\{1,2,3\}$ and $B=\{4,5\}$ then $A\times B$ is given by $A\times B=\{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$

**Relation** A relation R from set A to B is defined as subset of $A\times B$ .

Example: Let A={1,2,3} and B={2,3,4,5}. Let R={(1,2),(1,5),(3,5)}. As $R\subset A\times B$ therefore R is a relation from A to B.

**Types of Relations**:

(1) **Reflexive**: A relation R from A to B is said to be reflexive if $(a,a)\in A\times B$ for all $a\in A$

**(2) Symmetric:** A relation R from A to B is said to be symmetric if $(a,b)\in A\times B\Rightarrow(b,a)\in A\times B$

**(3) Transitive:** A relation R from A to B is said to be transitive if $(a,b) \text{ and } (b,c)\in A\times B\Rightarrow(a,c)\in A\times B$

**Equivalence Relation:** A relation R from A to B is said be equivalence if and only if it is reflexive, symmetric and transitive.

**Example-1:** Let R be a relation in set of natural numbers N defined as $R=\{(a,b): a\leqslant b\}$ . Show that R is reflexive, transitive but not symmetric.

Solution: For any $a\in N$ we have $a\leqslant a$ . Thus $(a,a)\in N\times N$ for all $a\in A$ . Therefore R is reflexive.

As $2\leq 3$ therefore $(2,3)\in N\times N$ but $3>2$ therefore $(3,2)\notin N\times N$ .Hence R is not symmetric.

Let $(a,b) \text{ and } (b,c)\in N\times N$ then $a\leq b \text{ and } b\leq c\Rightarrow a\leq c\Rightarrow(a,c)\in N\times N$ . Hence R is transitive.

**Example 2:** Let R be a relation set of natural numbers $N$ defined as $R=\{(a,b): a|b\}$ . Show that R is reflexive and transitive but not symmetric.

Solution: For any $a\in N$ we have $a|a$ therefore $(a,a)\in N\times N\,\forall\, n\in N$ . Thus R is symmetric relation.

$(2,4)\in N\times N \text{ as } 2|4$ but $(4,2)\notin N\times N$ as 4 does not divide 2. Thus R is not symmetric.

Let $(a,b) \text{ and } (b,c)\in N\times N\Rightarrow a|b \text{ and } b|c$ $\Rightarrow b=k_1\times a$ and $c=k_2\times b$ for some natural numbers $k_1 \text{ and } k_2$ . Thus $c=k_2 b=k_2 k_1 a$ . As $k_1 \text{ and } k_2\in N$ therefore $k_1 k_2\in N$ . Thus $a|c\Rightarrow(a,c)\in N\times N$ Therefore R is transitive relation.

**Example 3:** Let A={1,2,3} and R is relation in set A defined as $R=\{(a,b): a+b \text{ is even}\}$ . Is R an equivalence relation in A.

Solution: $R=\{(1,1),(1,3),(2,2),(3,1),(3,3)\}$ As R is reflexive, symmetric and transitive therefore R is an equivalence relation in set A.

**Example 4:** Let R be relation in set of real numbers $\mathbb{R}$ defined as $R=\{(a,b): a\leq b^2\}$ . Then show that R is neither reflexive nor symmetric nor transitive.

**Solution:** As $\frac{1}{2}>\left(\frac{1}{2}\right)^2$ thus R is not reflexive.

$2\leq 5^2$ thus $(2,5)\in R$ but $5>2^2$ therefore $(5,2)\notin R$ . Thus R is not symmetric.

$(10,4) \text{ and } (4,3)\in R$ as $10\leq 4^2 \text{ and } 4\leq 3^2$ but $(10,3)\notin R \text{ as } 10>3^2$ thus R is not transitive.

**Example 5:** Let R be relation on set $N\times N$ defined as $(a,b)R(c,d)\Leftrightarrow a+d=b+c$ . Show that R is an equivalence relation.

Solution: For any $(a,b)\in N\times N$ we have $a+b=b+a\Rightarrow(a,b)R(a,b)\forall(a,b)\in N\times N$, therefore R is reflexive relation.
Let $(a,b)R(c,d)\Rightarrow a+d=b+c\Rightarrow c+b=d+a\Rightarrow(c,d)R(a,b)$, therefore R is symmetric relation.
Let $(a,b)R(c,d)$ and $(c,d)R(e,f)\Rightarrow a+d=b+c$ and $c+f=d+e$ Adding these two we get $a+d+c+f=b+c+d+e\Rightarrow a+f=b+e\Rightarrow(a,b)R(e,f)$ therefore R is transitive relation.
As R is reflexive, symmetric and transitive therefore R is an equivalence relation.

**Example 6:** Let R be relation in set $N$ of natural numbers defined as $R=\{(a,b):|a-b| \text{ is multiple of } 3\}$. Show that R is an equivalence relation.

Solution: For any $a\in N$ we have $|a-a|=0$ which is multiple of 3 thus $(a,a)\in R\,\forall a\in A$. Therefore R is reflexive relation.
Let $(a,b)\in R\Rightarrow|a-b| \text{ is multiple of } 3\Rightarrow|b-a|\Rightarrow(b,a)\in R$ $(\text{as } |b-a|=|a-b|)$
Thus R is symmetric relation.
Let $(a,b)$ and $(b,c)\in R\Rightarrow|a-b|$ and $|b-c|$ are multiples of $3\Rightarrow a-b=3k_1$ and $b-c=3k_2$ for some integers $k_1$ and $k_2$ therefore $a-c=a-b+b-c=3\times(k_1+k_2)\Rightarrow|a-c|$ is multiple of 3 $\Rightarrow(a,c)\in R$ therefore R is transitive relation.
As R is reflexive, symmetric and transitive therefore R is an equivalence relation.

**Example 7:** Show that number of equivalence relations on a set {1,2,3} containing (1,2) and (2,1) is two

Solution: The smallest equivalence relation $R_1$ containing (1,2) and (2,1) is
$R_1=\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$
Now, we are left with four pairs (2,3),(3,2),(1,3) and (3,1). If we add any one say, (2,3) to $R_1$ then for symmetry

**Example8**: Let R be relation in set $L$ of all lines in plane given by $R=\{(l_1,l_2):l_1\|l_2\}$. Show that R is equivalence relation.

Solution: For each line $l\in L$ we have $l\|l\Rightarrow(l,l)\in R\,\forall l\in L\Rightarrow R$ is reflexive.
Let $l_1,l_2\in L$ such that $(l_1,l_2)\in R$. Then $(l_1,l_2)\in R\Rightarrow l_1\|l_2\Rightarrow l_2\|l_1\Rightarrow(l_2,l_1)\in R$. Thus R is symmetric.
Let $l_1,l_2,l_3\in L$ such that $(l_1,l_2)$ and $(l_2,l_3)\in R$
$(l_1,l_2)$ and $(l_2,l_3)\in R\Rightarrow l_1\|l_2$ and $l_2\|l_3\Rightarrow l_1\|l_3\Rightarrow(l_1,l_3)\in R$, therefore R is transitive relation.
R being reflexive, symmetric and transitive is an equivalence relation on L

**Example9:** Let $R_1$ and $R_2$ are two equivalence relations on set $A$. Then show that $R_1\cap R_2$ is also an equivalence relation on set $A$.

**Solution:** As $R_1$ and $R_2$ are reflexive therefore $(a,a)\in R_1$ and $(a,a)\in R_2\,\forall a\in A\Rightarrow(a,a)\in R_1\cap R_2$ $\forall a\in A$. Thus $R_1\cap R_2$ is also reflexive relation in set $A$.
Let $(a,b)\in R_1\cap R_2\Rightarrow(a,b)\in R_1$ and $(a,b)\in R_2$ as $R_1$ and $R_2$ are symmetric
$\Rightarrow(b,a)\in R_1$ and $(b,a)\in R_2\Rightarrow(b,a)\in R_1\cap R_2$
Thus $R_1\cap R_2$ is also symmetric.
Let $(a,b)$ and $(a,c)\in R_1\cap R_2\Rightarrow\{(a,b)$ and $(a,c)\in R_1\}$ and $\{(a,b)$ and $(a,c)\in R_2\}$ As $R_1$ and $R_2$ are transitive therefore $\Rightarrow(a,c)\in R_1$ and $(a,c)\in R_2\Rightarrow(a,c)\in R_1\cap R_2$. Thus $R_1\cap R_2$ is also transitive relation.
$R_1\cap R_2$ being reflexive, symmetric and transitive is an equivalence relation in set $A$

**Example-10:** Show that union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

Proof: Let $A=\{a,b,c\}$ and let R and S be two relations on A given by

$$R=\{(a,a),(b,b),(c,c),(a,b),(b,a)\}$$

and, $$S=\{(a,a),(b,b),(c,c),(b,c),(c,b)\}$$

It can be easily proved that R and S both are equivalence relations on set A. But $R\cup S$ is not transitive as $(a,b)\in R\cup S$ and $(b,c)\in R\cup S$ but $(a,c)\notin R\cup S$

Hence $R\cup S$ is not an equivalence relation.

## Exercise 1.1

1. Let R be the relation set N of natural numbers defined by $R=\{(a,b):a\geq b\}$ . Show that R is reflexive, symmetric but not transitive.
2. Let R be relation on set N of natural numbers defined as $R=\{(a,b):a\times b \text{ is even}\}$ . Is R an equivalence relation.
3. Let R be relation on set of all lines in a plane given by $R=\{(l_1,l_2):l_1\perp l_2\}$ . Is *R* an equivalence relation.
4. Let R be relation on set Z of integers defined as $R=\{(a,b):a-b \text{ is divisible by } 3\}$ Show that R is equivalence relation.
5. Let R be relation on set $N\times N$ defined as $(a,b)\,R\,(c,d)\Leftrightarrow ad=bc$ . Show that R is an equivalence relation.
6. Let $A=\{1,2,3\}$ . Find the number of equivalence relations on set A containing $(2,3)$ and $(3,2)$
7. Let R be relation on $P(A)$ ( Power set of A where A is non empty) defined as $R=\{(X,Y):X\subset Y\}$ . Is R an equivalence relation on set of A.
8. Let R be a relation on set $A=\{1,2,3\}$ given by $R=\{(1,1),(2,2),(3,3),(1,4)\{$ . Is R equivalence relation?
9. Let A be set of all triangles in a plane and let R be relation on set A defined as $R=\{(T_1,T_2):T_1\simeq T_2\}$ . Show that R is an equivalence relation on set A.
10. Let A be set of all human beings in a particular locality and let R be relation on set A defined as $R=\{(x,y): \text{ x is wife of y}\}$ . Show that R is neither reflexive nor symmetric but transitive.
11. Let R be relation on set of all points in a plane defined as $R=\{(a,b): \text{distance b/w aand b is less than } 5\}$ . Is R an equivalence relation.
12. Check whether the relation R defined on the set $A=\{1,2,3,4,5,6\}$ as $R=\{(a,b):b=a+1\{$ is reflexive, symmetric or transitive.
13. Show that the relation R defined on set ℝ of real numbers, defined as $R=\{(a,b):a\leq b^2\}$ is neither reflexive, nor symmetric nor transitive.
14. Check whether the relation R defined on set ℝ of real numbers, as $R=\{(a,b):ab\geq 0\}$ is reflexive, symmetric or transitive.
15. Show that the relation R on set of all books in a library of a college, given by $R=\{(a,b):a \text{ and } b \text{ have the same number of pages}\}$ is an equivalence relation set A.
16. Let N denote the set of all natural numbers and R be relation on set $N\times N$ defined by $(a,b)\,R\,(c,d)\Leftrightarrow ad(b+c)=bc(a+d)$ . Show that R is an equivalence relation on $N\times N$
17. Check whether the relation R defined on set $\mathbb{Z}$ of integers as $R=\{(a,b):a|b\}$ is reflexive, symmetric or transitive relation.
18. Let R be the relation on set $\mathbb{Z}$ of integers defined as $R=\{(a,b):a-b \text{ is multiple of } 3\}$ . Show that R is an equivalence relation set $\mathbb{Z}$ . Find the set of all elements related to integer zero.